SOLITON RESOLUTION FOR THE ENERGY-CRITICAL NONLINEAR HEAT EQUATION IN THE RADIAL CASE

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Abstract. We establish the Soliton Resolution Conjecture for the radial critical non-linear heat equation in dimension $D \geq 3$. Thus, every finite energy solution resolves, continuously in time, into a finite superposition of asymptotically decoupled copies of the ground state and free radiation.

1. Introduction

1.1. Setting of the Problem.

We study the energy-critical semi-linear heat flow on $\mathbb{R}^D$,

$$\begin{aligned} \partial_t u &= \Delta u + |u|^{\frac{4}{D-2}} u \\ u(0,x) &= u_0(x), \end{aligned} \tag{1.1}$$

where $D \geq 3$, $u = u(t,r) \in \mathbb{R}$, where $r = |x| \in (0,\infty)$ is the radial coordinate in $\mathbb{R}^D, \Delta := \partial_r^2 + (D-1)r^{-1}\partial_r$ is the radial Laplacian in $\mathbb{R}^D$. We restrict ourselves to smooth solutions that remain uniformly bounded in the energy space $\mathcal{E}$, i.e.

$$\|u\|_{\mathcal{E}}^2 := \int_0^\infty \left[(\partial_r u(r))^2 + \frac{(u(r))^2}{r^2} \right] r^{D-1} \, \mathrm{d}r < +\infty.$$

By [BC96], given finite energy data we let $T_+ > 0$ denote the maximal forward time of existence. Define the nonlinear conserved energy functional associated with (1.1)

$$E(u) := \int_0^\infty \frac{1}{2} (\partial_r u(r))^2 r^{D-1} \, \mathrm{d}r - \int_0^\infty \frac{D-2}{2D} |u(r)|^{\frac{2D}{D-2}} r^{D-1} \, \mathrm{d}r$$

along with the energy density

$$\mathbf{e}(u(r), r) = \frac{1}{2}(\partial_r u)^2 - \frac{D-2}{2D} |u(r)|^{\frac{2D}{D-2}}. \tag{1.2}$$

Solutions to (1.1) are invariant under the scaling

$$u(t,r) \mapsto u_\lambda(t,r) := \lambda^{-\frac{D-2}{2}} u\left(t/\lambda^2, r/\lambda\right), \quad \lambda > 0$$

and (1.1) is called energy-critical since $E(u) = E(u_\lambda)$. Testing (1.1) by $\partial_t u$ and integrating by parts we observe the formal energy identity

$$E(u(T)) + \int_0^T \|\mathcal{T}(u(t))\|_{L^2}^2 \, \mathrm{d}t = E(u(0))$$

for each $T > 0$, where $\mathcal{T}(u) := \partial_r^2 u + \frac{D-1}{r}\partial_r u + |u|^{\frac{4}{D-2}} u$. We define the Aubin-Talenti elliptic solution, $W : \mathbb{R}^D \to \mathbb{R}$, by

$$W(x) := \left(1 + \frac{|x|^2}{D(D-2)}\right)^{-\frac{D-2}{2}}.$$

and recall that by Caffarelli-Gidas-Spruck [CGS89], all entire positive solutions to the stationary equation

$$-\Delta W(x) = |W(x)|^{\frac{4}{D-2}} W(x), \quad x \in \mathbb{R}^D$$

are given by Aubin-Talenti bubbles up to sign, scaling, and translation. Since the elliptic solutions W are radially symmetric, we will often denote them $W(x) = W(r)$ with $r = |x|$. For each $\lambda > 0$, we write $W_\lambda(r) := \lambda^{-\frac{D-2}{2}} W(r/\lambda)$.

1.2. Statement of the Main result.

Theorem 1.1 (Soliton Resolution)**.** *Let $D \geq 3$ and let $u(t)$ be a finite energy solution to* (1.1) *with initial data $u(0) = u_0 \in \mathcal{E}$, defined on its maximal forward interval of existence $[0, T_+)$. Suppose that,*

$$\limsup_{t \to T_+} \|u(t)\|_{\mathcal{E}} < \infty.$$

Then either

(i) *$T_+ = \infty$, there exist a time $T_0 > 0$, an integer $N \geq 0$, continuous functions $\lambda_1(t), \ldots, \lambda_N(t) \in C^0([T_0, \infty))$, signs $\iota_1, \ldots, \iota_N \in \{-1, 1\}$, and $g(t) \in \mathcal{E}$ defined by*

$$u(t) = \sum_{j=1}^{N} \iota_j W_{\lambda_j(t)} + g(t),$$

such that

$$\|g(t)\|_{\mathcal{E}} + \sum_{j=1}^{N} \frac{\lambda_j(t)}{\lambda_{j+1}(t)} \to 0 \text{ as } t \to \infty,$$

where above we use the convention that $\lambda_{N+1}(t) = \sqrt{t}$;

(ii) *$T_+ < \infty$, there exists a time $T_0 < T_+$, a function $u^* \in \mathcal{E}$, an integer $N \geq 1$, continuous functions $\lambda_1(t), \ldots, \lambda_N(t) \in C^0([T_0, T_+))$, signs $\iota_1, \ldots, \iota_N \in \{-1, 1\}$, and $g(t) \in \mathcal{E}$ defined by*

$$u(t) = \sum_{j=1}^{N} \iota_j W_{\lambda_j(t)} + u^* + g(t),$$

such that

$$\|g(t)\|_{\mathcal{E}} + \sum_{j=1}^{N} \frac{\lambda_j(t)}{\lambda_{j+1}(t)} \to 0 \text{ as } t \to T_+,$$

where above we use the convention that $\lambda_{N+1}(t) = \sqrt{T_+ - t}$.

Remark 1.2. In the parabolic setting, similar results have been established for the harmonic map heat flow by Jendrej, Lawrie and Schlag in [JL23a, JLS23] following the bubbling theory for harmonic maps pioneered by Struwe [Str85] which was further developed in [Top97, Top04, Str85, QT97, Qin95]. For the nonlinear heat equation, such a result has been conjectured to be true and, during the preparation of this paper, was also raised as an open question by Kim and Merle in [KM24].

Remark 1.3. The Soliton Resolution Conjecture states that the evolution of generic solutions to nonlinear dispersive equations decouple as a sum of modulated traveling waves (or Solitons) and a free radiation term. The conjecture arose in the 1970s following the numerical simulations of Fermi–Pasta–Ulam [FPU55], Zabusky–Kruskal [ZK65]. Following the breakthrough work of Kenig and Merle [KM06], the Soliton Resolution Conjecture has been established for some classes of dispersive PDEs. In particular, for the wave equation the Soliton resolution conjecture has been proved in radial case for either odd space dimensions or $D = 4, 6$ in [DKM12, DKM13, DKM23, DKMM22, CDKM22] using the method of energy channels. On the other hand, using virial inequalities, Jia and Kenig in [JK17] proved the sequential soliton resolution in dimension $D = 6$. Recently, Jendrej and Lawrie in [JL23b] established the Soliton Resolution Conjecture for the radial energy critical nonlinear wave equation in all space dimensions $D \geq 4$ using a novel argument based on the analysis of collision intervals. They also established the same result for the equivariant energy critical wave maps equation in [JL22]. Finally, the Soliton Resolution Conjecture has also been established for the equivariant self-dual Chern–Simons–Schrödinger equation in [KKO23].

Remark 1.4. The nonlinear heat equation with power nonlinearity

$$\begin{cases} \partial_t u - \Delta u = |u|^{p-1}u & \text{in } \mathbb{R}^n \times (0,T) \\ u(x,0) = u_0(x) \end{cases}$$

has been the subject of intensive study beginning with the seminal works of Giga and Kohn [GK87, GMS04]. For a detailed introduction see the excellent monograph of Quittner and Souplet [QS19]. Since our main theorem deals with the asymptotic behavior near the blow-up time, we briefly review some results in that direction. When the nonlinearity is energy subcritical, i.e. $1 < p < +\infty$ when $n = 1, 2$ and $1 < p < \frac{n+2}{n-2}$ when $n \geq 3$ then [GK87, GMS04] showed that any blow-up solution is of Type I, i.e. there exists a constant $C > 0$ such that

$$\|u\|_{L^\infty(\mathbb{R}^n)} \leq C(T_+ - t)^{-\frac{1}{p-1}}.$$

Otherwise, the blow-up is of Type II. In the energy critical setting, i.e. $p = \frac{n+2}{n-2}$, Filippas, Herrero and Velázquez [FHV00], established that the solution exhibits Type I blow-up if the initial data is positive and radially decreasing and the dimension $D \geq 3$. Next, assuming that the initial datum is close to the ground state, Collot, Merle and Raphael in [CMR17] established a Trichotomy in dimension $D \geq 7$; the solution either dissipates to zero, or approaches to a rescaled Aubin–Talenti solution, or blows up in finite time and the blow-up is of Type I. Thus, if the initial data is close to a ground state in dimension $D \geq 7$, the solution does not exhibit Type II blow-up. Recently Wang and Wei in [WW21] established that for positive initial datum in $L^\infty(\mathbb{R}^n)$ and $D \geq 7$, any blow-up solution is of Type I. In contrast to the previous results, Theorem 1.1 is concerned with finite energy solutions that either exist globally in time or exhibit Type II blow-up. Examples of such solutions exhibiting non-trivial bubble tree decompositions have been recently constructed for the critical nonlinearity in dimensions $D \geq 7$ by del Pino, Musso, and Wei in [dPMW21]. Furthermore, the recent work of Kim and Merle [KM24] shows that under the assumption of radial symmetry, the above bubble tree constructions are the only possible examples in dimensions $D \geq 7$.

1.3. **Summary of the Proof.** Our proof is inspired by the recent breakthrough works of Jendrej–Lawrie, in particular, [JL23a] where they established an analogous version of Theorem 1.1 for the harmonic map heat flow in the equivariant setting. Before, going into more details we first make a few definitions.

Definition 1.5 (Multi-bubble configuration)**.** Given $M \in \{0, 1, \ldots\}$, $\vec{\iota} = (\iota_1, \ldots, \iota_M) \in \{-1, 1\}^M$ and an increasing sequence $\vec{\lambda} = (\lambda_1, \ldots, \lambda_M) \in (0, \infty)^M$, a multi-bubble configuration is defined by the formula

$$\mathcal{W}(\vec{\iota}, \vec{\lambda}; r) := \sum_{j=1}^{M} \iota_j W_{\lambda_j}(r).$$

When $M=0$, $\mathcal{W}(\vec{\iota}, \vec{\lambda}; r) := 0$.

With this definition, we define a localized distance function to multi-bubble configurations by

$$\delta_R(\boldsymbol{u}) := \inf_{M, \vec{\lambda}, \vec{\lambda}} \left(\|u - \mathcal{W}(\vec{\iota}, \vec{\lambda})\|^2_{\mathcal{E}(r \le R)} + \sum_{j=1}^{M} \left(\frac{\lambda_j}{\lambda_{j+1}} \right)^{\frac{D-2}{2}} \right)^{\frac{1}{2}} \tag{1.3}$$

where the infimum is taken over all $M \in \{0, 1, 2, \ldots\}$, all vectors $\iota \in \{-1, 1\}^M, \vec{\lambda} \in (0, \infty)^M$, and we use the convention that the last scale $\lambda_{M+1} = R$.
The first step in the Jendrej–Lawrie framework is to establish a localized sequential compactness lemma, which essentially states that a sequence of maps with vanishing tension converge (locally in space) to a multi-bubble configuration. Thus, given a sequence of maps u_n with bounded energy, a sequence $\rho_n \in (0, \infty)$ of scales, and tension $\mathcal{T}$ vanishing in L^2 relative to the scale ρ_n, i.e., $\lim_{n\to\infty} \rho_n \|\mathcal{T}(u_n)\|_{L^2} = 0$, there exists a subsequence of the u_n that converges to a multi-bubble configuration up to large scales relative to ρ_n, i.e., $\lim_{n\to\infty} \boldsymbol{\delta}_{R_n \rho_n}(u_n) = 0$ for some sequence $R_n \to \infty$. Fortunately, sequential compactness for the nonlinear critical heat equation has been established recently in [Law23].

Lemma 1.6 (Localized sequential bubbling)**.** *Let $u(t)$ be the solution to* (1.1) *with smooth initial data $u(0) = u_0 \in \mathcal{E}$, defined on its maximal interval of existence* $[0, T_+)$*. Suppose that*

$$\limsup_{t \to T_+} \|u(t)\|_{\mathcal{E}} < \infty.$$

Then either
(i) *$T_+ = \infty$, there exist a sequence of times $t_n \to \infty$, and a sequence $R_n \to \infty$ such that,*

$$\lim_{n\to\infty} \delta_{R_n \sqrt{t_n}} \left(u\left(t_n\right) \right) = 0.$$

(ii) *If $T_+ < \infty$, there exist a sequence of times $t_n \to T_+$, and a sequence $R_n \to \infty$ such that,*

$$\lim_{n\to\infty} \delta_{R_n \sqrt{T_+ - t_n}} \left(u\left(t_n\right) \right) = 0.$$

The proof of the above result is a consequence of the following lemma, which we recall below since it will used in the final section of the proof.

Lemma 1.7 (Compactness Lemma)**.** *Let $u_n \in \mathcal{E}$ be a sequence with* $\limsup_{n\to\infty} \|u_n\|_{\mathcal{E}} < \infty$*. Let $\rho_n \in (0, \infty)$ be a sequence and suppose that*

$$\lim_{n\to\infty} \left(\rho_n \left\| \mathcal{T}\left(u_n\right) \right\|_{L^2} \right) = 0.$$

Then, there exists a sequence $R_n \to \infty$ so that, up to passing to a subsequence of the u_n, we have,

$$\lim_{n\to\infty} \delta_{R_n \rho_n} \left(u_n\right) = 0.$$

The subsequence u_n can be chosen so that there are fixed $(M, \vec{\iota}) \in \mathbb{N} \cup \{0\} \times \{-1, 1\}^M$, a sequence $\vec{\lambda}_n \in (0, \infty)^M$, and $C_0 > 0$ with

$$\lim_{n\to\infty} \left(\| u_n - \mathcal{W}\left(\vec{\iota}, \vec{\lambda}_n\right) \|_{\mathcal{E}(r \le R_n \rho_n)} + \sum_{j=1}^{M-1} \left(\frac{\lambda_{n,j}}{\lambda_{n,j+1}} \right)^{\frac{D-2}{2}} \right) = 0,$$

and,

$$\lambda_{n,M} \le C_0 \rho_n, \quad \forall n.$$

Define $\mathbf{d}(t)$ to be the distance to the N-bubble configuration obtained from the compactness lemma (see Definition 5.1). Lemma 1.6 implies that

$$\lim_{n\to\infty} \mathbf{d}(t_n) = 0.$$

Theorem 1.1 follows from the fact that $\lim_{t\to\infty} \mathbf{d}(t) = 0$. To prove this we argue by contradiction, i.e. suppose that $\mathbf{d}(t)$ does not converge to zero. Then heuristically this means that $\mathbf{d}(t)$ is large along a certain sequence of times or in other words, $u(t)$ deviates from the N-bubble configuration. Instead of analyzing this sequence of times, the key insight in [JL22] is to consider a sequence of time intervals that keeps track of the dynamical history of the N-bubble configuration. Thus consider $[a_n, b_n]$, a sequence of time intervals where near the endpoints a_n and b_n, $u(t)$ is close N-bubble configuration while inside $[a_n, b_n]$, $u(t)$ is close to $(N-K)$-bubbles, where we choose K to be smallest non-negative integer minimal with respect to the above properties. Observe that intuitively, $K \geq 1$ since otherwise, $u(t)$ will always be close to the N bubble configuration which would imply that $\lim_{t\to\infty} \mathbf{d}(t) = 0$. Next, by using modulation theory on these collision intervals we can derive differential inequalities for scales of the bubbles that come into collision. Let λ_K denote the scale of K-th bubble that loses its shape, i.e. it undergoes collision on $[a_n, b_n]$. Then following [JL23a] we can show that for n large enough, there exists $[c_n, d_n] \subset [a_n, b_n]$ such that $d_n - c_n \gtrsim n^{-1}\lambda_K(c_n)^2$. Combined with Lemma 1.6 this implies that $\inf_{t\in[c_n,d_n]} \lambda_K(c_n)^2 \|\mathcal{T}(u(t)\|_{L^2}^2 \gtrsim 1$. Thus using the fact that $\int_0^\infty \|\mathcal{T}(u(t))\|_{L^2}^2 < +\infty$ we get that

$$C \geq \int_0^\infty \|\mathcal{T}(u(t))\|_{L^2}^2 \,\mathrm{d}t \geq \sum_{n\in\mathbb{N}} \int_{c_n}^{d_n} \|\mathcal{T}(u(t))\|_{L^2}^2 \,\mathrm{d}t \gtrsim \sum_{n\in\mathbb{N}} n^{-1} = +\infty,$$

which is a contradiction and therefore $\lim_{t\to\infty} \mathbf{d}(t) = 0$.

1.4. **Notational Conventions.** Given a function $\phi(r)$ and $\lambda > 0$, we denote by $\phi_\lambda(r) = \lambda^{-\frac{D-2}{2}}\phi(r/\lambda)$, the H-invariant rescaling, and by $\phi_{\underline{\lambda}}(r) = \lambda^{-\frac{D}{2}}\phi(r/\lambda)$ the L^2-invariant rescaling. We denote by $\Lambda := r\partial_r + \frac{D-2}{2}$ and $\underline{\Lambda} := r\partial r + \frac{D}{2}$ the infinitesimal generators of these scalings. Next, we denote the nonlinear energy by E and the energy space by $\mathcal{E}$. We use the notation $\widetilde{E}(r_1, r_2)$ or $\mathcal{E}(r_1, r_2)$ to denote the local energy norm

$$\widetilde{E}(r_1, r_2) := \|g\|_{\mathcal{E}(r_1,r_2)}^2 := \int_{r_1}^{r_2} \left((\partial_r g)^2 + \frac{g^2}{r^2}\right) r^{D-1}\mathrm{d}r,$$

By convention, $\mathcal{E}(r_0) := \mathcal{E}(r_0, \infty)$ for $r_0 > 0$. Similarly, we denote the local nonlinear energy as $E(u; r_1, r_2)$. The notation $A \lesssim B$ means that $A \leq CB$ and $A \gtrsim B$ means that $A \geq cB$ for some constants $c > 0, C > 0$ possibly depending on the number of bubbles N. We write $A \ll B$ if $\lim_{n\to\infty} A/B = 0$.

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2. Preliminaries

2.1. **Local Cauchy Theory.** We first recall the local well-posedness theory for the heat equation in the energy space. See for instance Theorem 2.1 in [GR18] or Proposition 2.1 in [CMR17].

Lemma 2.1 (Local well-posedness). *For $u_0 \in \mathcal{E}$ there exists a maximal time of existence $T_+ = T_+(u_0)$ and a unique solution $u(t) \in \mathcal{E}$ to (1.1) on the time interval $t \in [0, T_+)$with $u(0) = u_0$. The maximal time is characterized by the following condition: if $T_+ < \infty$ then*

$$\lim_{t \to T_+} \|u(t)\|_{L^\infty} = +\infty.$$

If there is no such $T_+ < \infty$, we say $T_+ = \infty$ and the flow is globally defined. The energy $E(u(t))$ is absolutely continuous and non-increasing as a function of $t \in [0, T]$ for any $T < T_+$, and for any $t_1 \leq t_2 \in [0, T_+)$, there holds,

$$E(u(t_2)) + \int_{t_1}^{t_2} \int_0^\infty (\partial_t u(t,r))^2 r^{D-1} \, \mathrm{d}r \, \mathrm{d}t = E(u(t_1)).$$

In particular,

$$\int_0^{T_+} \int_0^\infty (\partial_t u(t,r))^2 r^{D-1} \, \mathrm{d}r \, \mathrm{d}t \lesssim \sup_{t \in [0, T_+)} \|u(t)\|_{\mathcal{E}}^2 < +\infty. \tag{2.1}$$

2.2. **Multi-Bubble Configuration.** Next, we recall some facts about solutions near multi-bubble configuration. For proofs and further references see [JL23b]. Denote $\mathcal{L}_{\mathcal{W}}$ the operator obtained by linearization of (1.1) about an M-bubble configuration $\mathcal{W}(\vec{\iota}, \vec{\lambda})$

$$\mathcal{L}_{\mathcal{W}}\, g := \mathrm{D}^2 E(\mathcal{W}(\vec{\iota}, \vec{\lambda}))g = -\Delta g - f'(\mathcal{W}(\vec{\iota}, \vec{\lambda}))g,$$

where $f(z) := |u|^{\frac{4}{D-2}} u$ and $f'(z) = \frac{D+2}{D-2} |z|^{\frac{4}{D-2}}$. Given $g \in \mathcal{E}$,

$$\left\langle \mathrm{D}^2 E(\mathcal{W}(\vec{\iota}, \vec{\lambda}))g \mid g \right\rangle = \int_0^\infty \Big((\partial_r g(r))^2 - f'(\mathcal{W}(\vec{\iota}, \vec{\lambda}))g(r)^2 \Big) r^{D-1} \mathrm{d}r.$$

Denote $\mathcal{L}_\lambda := -\Delta - f'(W_\lambda)$ the linearization around a single bubble, W_λ and set $\mathcal{L} := \mathcal{L}_1$. Next, define the infinitesimal generators of $\dot{H}^1$-invariant dilations by Λ and in the L^2-invariant case we write $\underline{\Lambda}$

$$\Lambda := r\partial_r + \frac{D-2}{2}, \quad \underline{\Lambda} := r\partial_r + \frac{D}{2}$$

Thus for instance,

$$\Lambda W(r) = \Big(\frac{D-2}{D} - \frac{r^2}{2D} \Big) \Big(1 + \frac{r^2}{D(D-2)} \Big)^{-\frac{D}{2}}$$

Regarding the spectrum of $\mathcal{L}$, [DM08, Proposition 5.5] showed that there exists a unique negative simple eigenvalue that we denote by $-\kappa^2 < 0$ (where $\kappa > 0$). Denote $\mathcal{Y}$ as the normalized (in L^2) eigenfunction associated to this eigenvalue. Fix any non-negative function $\mathcal{Z} \in C_0^\infty(0, \infty)$ such that the following holds

$$\langle \mathcal{Z} \mid \Lambda W \rangle > 0 \quad \text{and} \quad \langle \mathcal{Z} \mid \mathcal{Y} \rangle = 0.$$

We record the following localized coercivity lemma, see [Jen19, Lemma 3.8].

Lemma 2.2. *There exist uniform constants $c < 1/2, C > 0$ with the following properties. Let $g \in \mathcal{E}$. Then,*

$$\langle \mathcal{L} g \mid g \rangle \geq c \|g\|_{\mathcal{E}}^2 - C \langle \mathcal{Z} \mid g \rangle^2 - C \langle \mathcal{Y} \mid g \rangle^2$$

If $R > 0$ is large enough then,

$$(1-2c)\int_0^R (\partial_r g(r))^2\, r^{D-1}\mathrm{d}r + c\int_R^\infty (\partial_r g(r))^2\, r^{D-1}\mathrm{d}r - \int_0^\infty f'(W(r))g(r)^2\, r^{D-1}\,\mathrm{d}r$$
$$\geq -C\,\langle \mathcal{Z} \mid g\rangle^2 - C\,\langle \mathcal{Y} \mid g\rangle^2 .$$

If $r > 0$ is small enough, then

$$(1-2c)\int_r^\infty (\partial_r g(r))^2\, r\mathrm{d}r + c\int_0^r (\partial_r g(r))^2\, r\mathrm{d}r - \int_0^\infty f'(W(r))g(r)^2\, r^{D-1}\,\mathrm{d}r$$
$$\geq -C\,\langle \mathcal{Z} \mid g\rangle^2 - C\,\langle \mathcal{Y} \mid g\rangle^2 .$$

Following the argument for instance outlined in the proof of Lemma 3.9 in [Jen19] we have the following coercivity property of $\mathcal{L}_{\mathcal{W}}$.

Lemma 2.3. *Let $M \in \mathbb{N}$. There exist $\eta, c_0 > 0$ with the following properties. Consider the subset of M-bubble configurations $\mathcal{W}(\vec{\iota}, \vec{\lambda})$ for $\vec{\iota} \in \{-1, 1\}^M$, $\vec{\lambda} \in (0,\infty)^M$ such that,*

$$\sum_{j=1}^{M-1}\Big(\frac{\lambda_j}{\lambda_{j+1}}\Big)^{\frac{D-2}{2}} \leq \eta^2. \tag{2.2}$$

Let $g \in \mathcal{E}$ be such that

$$0 = \big\langle \mathcal{Z}_{\underline{\lambda_j}} \mid g\big\rangle \quad for \quad j = 1, \dots M$$

for $\vec{\lambda}$ as in (2.2). Then,

$$\big\langle \mathcal{L}_{\mathcal{W}} g \mid g\big\rangle + \sum_{j=1}^{M}\big\langle \mathcal{Y}_{\lambda_j} \mid g\big\rangle^2 \geq c_0\|g\|_{\mathcal{E}}^2.$$

Next arguing as in Lemma 2.22 in [JL22] we obtain the following energy expansion around a multi-bubble configuration.

Lemma 2.4. *Let $M \in \mathbb{N}$. For any $\theta > 0$, there exists $\eta > 0$ with the following property. Consider the subset of M-bubble configurations $\mathcal{W}(\iota, \vec{\lambda})$such that*

$$\sum_{j=1}^{M-1}\Big(\frac{\lambda_j}{\lambda_{j+1}}\Big)^{\frac{D-2}{2}} \leq \eta.$$

Then,

$$\Big|E(\mathcal{W}(\vec{\iota}, \vec{\lambda})) - ME(W) + \frac{(D(D-2))^{\frac{D}{2}}}{D}\sum_{j=1}^{M-1}\iota_j\iota_{j+1}\Big(\frac{\lambda_j}{\lambda_{j+1}}\Big)^{\frac{D-2}{2}}\Big| \leq \theta\sum_{j=1}^{M-1}\Big(\frac{\lambda_j}{\lambda_{j+1}}\Big)^{\frac{D-2}{2}}.$$

Moreover, there exists a uniform constant $C > 0$ such that for any $g \in \mathcal{E}$,

$$\Big|\Big\langle DE(\mathcal{W}(m, \vec{\iota}, \vec{\lambda})) \mid g\Big\rangle\Big| \leq C\|g\|_{\mathcal{E}}\sum_{j=1}^{M}\Big(\frac{\lambda_j}{\lambda_{j+1}}\Big)^{\frac{D-2}{2}}.$$

To measure how much we deviate from a multi-bubble configuration, we define the following proximity function.

Definition 2.5. Fix M as in Definition 1.5 and let $v \in \mathcal{E}$. Define,

$$\mathbf{d}_M(v) := \inf_{\vec{\iota},\vec{\lambda}} \Big(\|v - \mathcal{W}(\vec{\iota},\vec{\lambda})\|_{\mathcal{E}}^2 + \sum_{j=1}^{M-1} \Big(\frac{\lambda_j}{\lambda_{j+1}}\Big)^{\frac{D-2}{2}} \Big)^{\frac{1}{2}}.$$

where the infimum is taken over all vectors $\vec{\lambda} = (\lambda_1, \dots, \lambda_M) \in (0,\infty)^M$ and all $\vec{\iota} = \{\iota_1, \dots, \iota_M\} \in \{-1,1\}^M$.

Using a similar argument as in Lemma 2.17 in [JL22], when the value of the proximity function is small and the energy is close to the sum of energies of each bubble then we can find signs $\vec{\iota}$ and scales $\vec{\lambda}$ that realize the infimum in the above definition

Lemma 2.6. *Let $M \in \mathbb{N}$. There exists $\eta, C > 0$ with the following properties. Let $\theta > 0$, and let $v \in \mathcal{E}$ be such that*

$$\mathbf{d}_M(v) \le \eta, \quad \text{and} \quad E(v) \le ME(W) + \theta^2.$$

Then, there exists a unique choice of $\vec{\lambda} = (\lambda_1, \dots, \lambda_M) \in (0,\infty)^M$, $\vec{\iota} \in \{-1,1\}^M$, and $g \in \mathcal{E}$, such that

$$v = \mathcal{W}(m, \vec{\iota}, \vec{\lambda}) + g, \quad 0 = \big\langle \mathcal{Z}_{\underline{\lambda_j}} \mid g \big\rangle, \quad \forall j = 1, \dots, M,$$

along with the estimates,

$$\mathbf{d}_M(v)^2 \le \|g\|_{\mathcal{E}}^2 + \sum_{j=1}^{M-1} \Big(\frac{\lambda_j}{\lambda_{j+1}}\Big)^{\frac{D-2}{2}} \le C\mathbf{d}_M(v)^2,$$

Defining the unstable component of g by (scaling ensures that $|a_j^-| \lesssim \|g\|_{\mathcal{E}}$),

$$a_j^- := \frac{\kappa}{\lambda_j} \langle \mathcal{Y}_{\underline{\lambda_j}} \mid g \rangle \tag{2.3}$$

we additionally have the estimates,

$$\|g\|_{\mathcal{E}}^2 + \sum_{j \notin \mathcal{S}} \Big(\frac{\lambda_j}{\lambda_{j+1}}\Big)^{\frac{D-2}{2}} \le C \max_{j \in \mathcal{S}} \Big(\frac{\lambda_j}{\lambda_{j+1}}\Big)^{\frac{D-2}{2}} + \max_{i \in \{1,\dots,M\}} |a_i^-|^2 + \theta^2, \tag{2.4}$$

where $\mathcal{S} := \{j \in \{1, \dots, M-1\} : \iota_j = \iota_{j+1}\}$.

Furthermore, similar to Lemma 2.20 in [JL22], if a function w is close to two different multi-bubble configurations then the scales of those two configurations are also the same up to a small constant.

Lemma 2.7. *There exists $\eta > 0$ sufficiently small with the following property. Let $M, L \in \mathbb{N}$, $\vec{\iota} \in \{-1,1\}^M, \vec{\sigma} \in \{-1,1\}^L$, $\vec{\lambda} \in (0,\infty)^M, \vec{\mu} \in (0,\infty)^L$, and $w = (w, 0)$ be such that $\|w\|_{\mathcal{E}} < \infty$ and,*

$$\|w - \mathcal{W}(\vec{\iota},\vec{\lambda})\|_{\mathcal{E}}^2 + \sum_{j=1}^{M-1} \Big(\frac{\lambda_j}{\lambda_{j+1}}\Big)^{\frac{D-2}{2}} \le \eta,$$

$$\|w - \mathcal{W}(\vec{\sigma},\vec{\mu})\|_{\mathcal{E}}^2 + \sum_{j=1}^{L-1} \Big(\frac{\mu_j}{\mu_{j+1}}\Big)^{\frac{D-2}{2}} \le \eta.$$

Then, $M = L$, $\vec{\iota} = \vec{\sigma}$. Moreover, for every $\theta > 0$ the number $\eta > 0$ above can be chosen small enough so that

$$\max_{j=1,\dots M} \Big| \frac{\lambda_j}{\mu_j} - 1 \Big| \le \theta.$$

Finally, since we linearize near a multi-bubble configuration, we will often need to compute the nonlinear interaction term coming from an M-bubble configuration, $\mathcal{W}(\vec{\iota},\vec{\lambda})$. Define the interaction term as

$$f_{\mathbf{i}}(\vec{\iota},\vec{\lambda}) := f(\mathcal{W}(\vec{\iota},\vec{\lambda})) - \sum_{j=1}^{M} \iota_j f(W_{\lambda_j}). \tag{2.5}$$

Then similar to Lemma 2.21 in [JL22] we have

Lemma 2.8. *Let $M \in \mathbb{N}$. For any $\theta > 0$, there exists $\eta > 0$ with the following property. Let $\mathcal{W}(\vec{\iota},\vec{\lambda})$ be an M-bubble configuration with*

$$\sum_{j=0}^{M} \Big(\frac{\lambda_j}{\lambda_{j+1}}\Big)^{\frac{D-2}{2}} \leq \eta,$$

under the convention that $\lambda_0 = 0$, $\lambda_{M+1} = \infty$. Then, we have,

$$\Big|\Big\langle \Lambda W_{\lambda_j} \mid f_{\mathbf{i}}(\vec{\iota},\vec{\lambda})\Big\rangle - \iota_{j-1}\frac{D-2}{2D}(D(D-2))^{\frac{D}{2}}\Big(\frac{\lambda_{j-1}}{\lambda_j}\Big)^{\frac{D-2}{2}} + \iota_{j+1}\frac{D-2}{2D}(D(D-2))^{\frac{D}{2}}\Big(\frac{\lambda_j}{\lambda_{j+1}}\Big)^{\frac{D-2}{2}}\Big| \leq \theta\Big(\Big(\frac{\lambda_{j-1}}{\lambda_j}\Big)^{\frac{D-2}{2}} + \Big(\frac{\lambda_j}{\lambda_{j+1}}\Big)^{\frac{D-2}{2}}\Big)$$

where here $f_{\mathbf{i}}(\vec{\iota},\vec{\lambda})$ is defined in (2.5).

2.3. **Localized Energy Inequalities and Energy Trapping.** Since our argument will rely on energy estimates, we record here some localized energy identities.

Definition 2.9 (Hardy Energy Density)**.** Denote the energy density associated with $\mathcal{E}$-norm as follows

$$\tilde{\mathbf{e}}(u) := (\partial_r u(r))^2 + \frac{u^2}{r^2}, \quad \widetilde{E}(u) := \int_0^\infty \tilde{\mathbf{e}}(u) r^{D-1} dr = \|u\|_{\mathcal{E}}^2.$$

Next, we will derive some monotonicity properties of this modified energy density $\tilde{\mathbf{e}}$.

Lemma 2.10. *Let $I \subset [0,\infty)$ be a time interval, and let $\phi : I \times (0,\infty) \to [0,\infty)$ be a smooth function. Let $u(t) \in \mathcal{E}$ be a solution to (1.1) on I. Then, for any $t_1 < t_2 \in I$,*

$$\begin{aligned}
&\int_{\mathbb{R}_+} \tilde{\mathbf{e}}(u(t_2))\phi^2 - \int_{\mathbb{R}_+} \tilde{\mathbf{e}}(u(t_1))\phi^2 \\
&= -2\int_{t_1}^{t_2}\int_{\mathbb{R}_+} (\partial_t u)^2\phi^2 + 2\int_{t_1}^{t_2}\int_{\mathbb{R}_+} |u|^{p-1}u(\partial_t u)\phi^2 - 4\int_{t_1}^{t_2}\int_{\mathbb{R}_+} (\partial_r u)(\partial_t u)\phi\partial_r\phi \\
&\quad + 2\int_{t_1}^{t_2}\int_{\mathbb{R}_+} \frac{u\partial_t u}{r^2}\phi^2 + 2\int_{t_1}^{t_2}\int_{\mathbb{R}_+} \tilde{\mathbf{e}}(u(t),r)\phi\partial_t\phi
\end{aligned} \tag{2.6}$$

If $\phi(t,r)$ satisfies, $\partial_t\phi(t,r) \leq 0$ for all $t \in [t_1,t_2]$ then,

$$\begin{aligned}
&\int_{\mathbb{R}_+} \tilde{\mathbf{e}}(u(t_2))\phi^2 - \int_{\mathbb{R}_+} \tilde{\mathbf{e}}(u(t_1))\phi^2 \\
&\leq -\int_{t_1}^{t_2}\int_{\mathbb{R}_+} (\partial_t u)^2\phi^2 + 4\int_{t_1}^{t_2}\int_{\mathbb{R}_+} |\partial_r u|^2|\partial_r\phi|^2 + 2\Big(\int_{t_1}^{t_2}\int_{\mathbb{R}_+} |u|^{2p}\phi^2\Big)^{1/2}\Big(\int_{t_1}^{t_2}\int_{\mathbb{R}_+} (\partial_t u)^2\phi^2\Big)^{1/2} \\
&\quad + 2\Big(\int_{t_1}^{t_2}\int_{\mathbb{R}_+} \frac{|u|^2}{r^4}\phi^2\Big)^{1/2}\Big(\int_{t_1}^{t_2}\int_{\mathbb{R}_+} (\partial_t u)^2\phi^2\Big)^{1/2}
\end{aligned} \tag{2.7}$$

and

$$\begin{aligned}
&\int_{\mathbb{R}_+} \tilde{\mathbf{e}}(u(t_2))\phi^2 - \int_{\mathbb{R}_+} \tilde{\mathbf{e}}(u(t_1))\phi^2 \\
&\leq -2\int_{t_1}^{t_2}\int_{\mathbb{R}_+} (\partial_t u)^2\phi^2 + 2\left(\int_{t_1}^{t_2}\int_{\mathbb{R}_+} |u|^{2p}\phi^2\right)^{1/2}\left(\int_{t_1}^{t_2}\int_{\mathbb{R}_+} (\partial_t u)^2\phi^2\right)^{1/2} \\
&\quad + 4\left(\int_{t_1}^{t_2}\int_{\mathbb{R}_+} (\partial_t u)^2\phi^2(\partial_r\phi)^2\right)^{1/2}\left(\int_{t_1}^{t_2}\int_{\mathbb{R}_+} (\partial_r u)^2\right)^{1/2} + 2\left(\int_{t_1}^{t_2}\int_{\mathbb{R}_+} \frac{|u|^2}{r^4}\phi^2\right)^{1/2}\left(\int_{t_1}^{t_2}\int_{\mathbb{R}_+} (\partial_t u)^2\phi^2\right)^{1/2}
\end{aligned} \tag{2.8}$$

Proof. By a standard density argument, it suffices to consider smooth and compactly supported solutions u. Observe that

$$\int_{\mathbb{R}_+} \tilde{\mathbf{e}}(u(t_2))\phi^2 - \int_{\mathbb{R}_+} \tilde{\mathbf{e}}(u(t_1))\phi^2 = \int_{t_1}^{t_2}\int_{\mathbb{R}_+} \partial_t(\tilde{\mathbf{e}}(u(t))\phi^2)dtr^{D-1}dr.$$

Thus we compute

$$\partial_t(\tilde{\mathbf{e}}(u(t))\phi^2) = 2\left(\partial_r u\partial_t\partial_r u\right)\phi^2 + \frac{2u\partial_t u}{r^2}\phi^2 + 2\tilde{\mathbf{e}}(u(t),r)\phi\partial_t\phi$$

which implies

$$\begin{aligned}
&\int_{\mathbb{R}_+} \tilde{\mathbf{e}}(u(t_2))\phi^2 - \int_{\mathbb{R}_+} \tilde{\mathbf{e}}(u(t_1))\phi^2 \\
&= 2\int_{t_1}^{t_2}\int_{\mathbb{R}_+} \left(\partial_r u\partial_t\partial_r u + \frac{u\partial_t u}{r^2}\phi^2\right)\phi^2 + 2\int_{t_1}^{t_2}\int_{\mathbb{R}_+} \tilde{\mathbf{e}}(u(t),r)\phi\partial_t\phi \\
&= -2\int_{t_1}^{t_2}\int_{\mathbb{R}_+} (\Delta u)\partial_t u\phi^2 - 4\int_{t_1}^{t_2}\int_{\mathbb{R}_+} (\partial_r u)(\partial_t u)\phi\partial_r\phi + 2\int_{\mathbb{R}_+} \frac{u\partial_t u}{r^2}\phi^2 + 2\int_{t_1}^{t_2}\int_{\mathbb{R}_+} \tilde{\mathbf{e}}(u(t),r)\phi\partial_t\phi \\
&= -2\int_{t_1}^{t_2}\int_{\mathbb{R}_+} (\partial_t u)^2\phi^2 + 2\int_{t_1}^{t_2}\int_{\mathbb{R}_+} |u|^{p-1}u(\partial_t u)\phi^2 - 4\int_{t_1}^{t_2}\int_{\mathbb{R}_+} (\partial_r u)(\partial_t u)\phi\partial_r\phi + 2\int_{\mathbb{R}_+} \frac{u\partial_t u}{r^2}\phi^2 \\
&\quad + 2\int_{t_1}^{t_2}\int_{\mathbb{R}_+} \tilde{\mathbf{e}}(u(t),r)\phi\partial_t\phi
\end{aligned}$$

Thus we have proved (2.6). To see (2.7) we use the assumption $\partial_t\phi \leq 0$ and the inequality $-4ab \leq a^2 + 4b^2$ with $a = \partial_t u\phi$, $b = \partial_r u\partial_r\phi$ to get

$$\begin{aligned}
&\int_{\mathbb{R}_+} \tilde{\mathbf{e}}(u(t_2))\phi^2 - \int_{\mathbb{R}_+} \tilde{\mathbf{e}}(u(t_1))\phi^2 \\
&= -2\int_{t_1}^{t_2}\int_{\mathbb{R}_+} (\partial_t u)^2\phi^2 + 2\int_{t_1}^{t_2}\int_{\mathbb{R}_+} |u|^{p-1}u(\partial_t u)\phi^2 - 4\int_{t_1}^{t_2}\int_{\mathbb{R}_+} (\partial_r u)(\partial_t u)\phi\partial_r\phi \\
&\quad + 2\int_{t_1}^{t_2}\int_{\mathbb{R}_+} \tilde{\mathbf{e}}(u(t),r)\phi\partial_t\phi + 2\int_{\mathbb{R}_+} \frac{u\partial_t u}{r^2}\phi^2 \\
&\leq -\int_{t_1}^{t_2}\int_{\mathbb{R}_+} (\partial_t u)^2\phi^2 + 2\left(\int_{t_1}^{t_2}\int_{\mathbb{R}_+} |u|^{2p}\phi^2\right)^{1/2}\left(\int_{t_1}^{t_2}\int_{\mathbb{R}_+} (\partial_t u)^2\phi^2\right)^{1/2} \\
&\quad + 4\int_{t_1}^{t_2}\int_{\mathbb{R}_+} |\partial_r u|^2|\partial_r\phi|^2 + 2\left(\int_{t_1}^{t_2}\int_{\mathbb{R}_+} \frac{|u|^2}{r^4}\phi^2\right)^{1/2}\left(\int_{t_1}^{t_2}\int_{\mathbb{R}_+} (\partial_t u)^2\phi^2\right)^{1/2}
\end{aligned}$$

where we used $\int_{t_1}^{t_2}\int_{\mathbb{R}_+}\tilde{\mathbf{e}}(u(t),r)\phi\partial_t\phi \le 0$ because $\partial_t\phi \le 0$. To see (2.8) we start with (2.6) and use Cauchy-Schwarz inequality for double integrals (see equation (1.20) in [Ste04])

$$
\begin{aligned}
&\int_{\mathbb{R}_+}\tilde{\mathbf{e}}(u(t_2))\phi^2 - \int_{\mathbb{R}_+}\tilde{\mathbf{e}}(u(t_1))\phi^2 \\
&= -2\int_{t_1}^{t_2}\int_{\mathbb{R}_+}(\partial_t u)^2\phi^2 + 2\int_{t_1}^{t_2}\int_{\mathbb{R}_+}|u|^{p-1}u(\partial_t u)\phi^2 - 4\int_{t_1}^{t_2}\int_{\mathbb{R}_+}(\partial_r u)(\partial_t u)\phi\partial_r\phi + 2\int_{t_1}^{t_2}\int_{\mathbb{R}_+}\frac{u\partial_t u}{r^2}\phi^2 \\
&\quad + 2\int_{t_1}^{t_2}\int_{\mathbb{R}_+}\tilde{\mathbf{e}}(u(t),r)\phi\partial_t\phi \\
&\le -2\int_{t_1}^{t_2}\int_{\mathbb{R}_+}(\partial_t u)^2\phi^2 + 2\left(\int_{t_1}^{t_2}\int_{\mathbb{R}_+}|u|^{2p}\phi^2\right)^{1/2}\left(\int_{t_1}^{t_2}\int_{\mathbb{R}_+}(\partial_t u)^2\phi^2\right)^{1/2} \\
&\quad + 4\left(\int_{t_1}^{t_2}\int_{\mathbb{R}_+}(\partial_t u)^2\phi^2(\partial_r\phi)^2\right)^{1/2}\left(\int_{t_1}^{t_2}\int_{\mathbb{R}_+}(\partial_r u)^2\right)^{1/2} + 2\left(\int_{t_1}^{t_2}\int_{\mathbb{R}_+}\frac{|u|^2}{r^4}\phi^2\right)^{1/2}\left(\int_{t_1}^{t_2}\int_{\mathbb{R}_+}(\partial_t u)^2\phi^2\right)^{1/2},
\end{aligned}
$$

which proves (2.8). □

Lemma 2.11 (Radial Sobolev Embedding)**.** *Let $v \in C_c^\infty(\mathbb{R}_+)$. Then for $R > 0$ we have*

$$|v(R)| \le \frac{\sqrt{2}}{R^{D-2}}\|v\|_{\mathcal{E}(R,\infty)}$$

Proof. Using the fact that v has compact support, we get

$$
\begin{aligned}
R^{(D-1)/2}v^2(R) &= -\int_R^\infty \partial_r(r^{(D-1)/2}v^2(r))dr \\
&= -\int_R^\infty\left(\frac{D-1}{2r}v^2(r) + 2v\partial_r v\right)r^{(D-1)/2}dr \\
&\le 2\int_R^\infty |v||\partial_r v|r^{(D-1)/2}dr \\
&\le 2\sqrt{\int_R^\infty |v|^2dr\int_R^\infty|\partial_r v|^2r^{(D-1)}dr} \\
&\le 2\sqrt{\frac{1}{R^{D-3}}\int_R^\infty\frac{v^2}{r^2}r^{D-1}dr\int_R^\infty|\partial_r v|^2r^{(D-1)}dr} \\
&\le \frac{2}{R^{(D-3)/2}}\|v\|^2_{\mathcal{E}(R,\infty)}
\end{aligned}
$$

and therefore

$$|v(R)| \le \frac{\sqrt{2}}{R^{D-2}}\|v\|_{\mathcal{E}(R,\infty)}.$$

□

The coercivity of the nonlinear energy E plays an important role in the proof of Theorem 1.1 since localized energy inequalities will imply smallness of the energy, which we would like to transfer to $\mathcal{E}$-norm to deduce smallness of $\mathcal{E}$-norm. This would in turn help us proving that distance function $\mathbf{d}(t)$ is small which is ultimately what we are after. Thus we would like to compare $E(u)$ with $\|u\|_{\mathcal{E}}$. This is not always possible, however when $\|u\|_{\mathcal{E}}$ is small then we can prove such an estimate. The following lemma is inspired by the energy trapping argument in [KM06].

Lemma 2.12 (Global Trapping)**.** *Let $v \in \mathcal{E}$ and let $\delta > 0$ such that*

$$\|v\|_{\mathcal{E}} \leq \delta$$

then there exists a constant $C = C(\delta) > 0$ such that

$$E(v) \geq C\|v\|_{\mathcal{E}}^2.$$

Proof. By Sobolev and Hardy's inequality, there exists constant $C_1, C_2 > 0$

$$\|\partial_r v\|_{L^2} \leq \|v\|_{\mathcal{E}} \leq (1 + C_1)\|\partial_r v\|_{L^2},$$
$$\|v\|_{L^{2^*}}^{2^*} \leq C_2\|\partial_r v\|_{L^2}^{2^*} \leq C_2\|v\|_{\mathcal{E}}^{2^*}.$$

Thus we get

$$\begin{aligned} E(v) &= \frac{1}{2}\int_{\mathbb{R}_+} |\partial_r v|^2 r^{D-1} dr - \frac{1}{2^*}\int_{\mathbb{R}_+} |v|^{2^*} r^{D-1} dr \\ &\geq \frac{1}{2(1 + C_1)}\|v\|_{\mathcal{E}}^2 - \frac{C_2}{2^*}\|v\|_{\mathcal{E}}^{2^*-2}\|v\|_{\mathcal{E}}^2 \\ &\geq \left(\frac{1}{2(1 + C_1)} - \frac{C_2}{2^*}\delta^{2^*-2}\right)\|v\|_{\mathcal{E}}^2 \\ &\geq C_\delta\|v\|_{\mathcal{E}}^2, \end{aligned}$$

for some constant $C_\delta > 0$ and $\delta > 0$ small enough. □

Unfortunately, we do not have smallness of $\mathcal{E}$-norm globally, but either up to some fixed scales or on tails. The following lemma shows that the previous lemma can be upgraded to deduce coercivity of the nonlinear energy when we have smallness of $\mathcal{E}$-norm only on the tails.

Lemma 2.13 (Trapping on Tails)**.** *Let $v \in \mathcal{E}$. There exist $\delta > 0$ and constant $C > 0$ such that for all $R > 0$ if*

$$\|v\|_{\mathcal{E}(R,\infty)} \leq \delta$$

then

$$E(v; R, \infty) \geq C\|v\|_{\mathcal{E}(R,\infty)}^2.$$

Proof. We first proved this statement for $v \in C_c^\infty(\mathbb{R}_+)$. The result then follows from a standard density/approximation argument. We want to show that there exists a constant $C > 0$ such that

$$E(v; R, \infty) \geq C\|v\|_{\mathcal{E}(R,\infty)}^2$$

which simplifies to

$$\frac{1}{2}\int_R^\infty |\partial_r v|^2 r^{D-1} dr - \frac{1}{2^*}\int_R^\infty |v|^{2^*} r^{D-1} dr \geq C\int_R^\infty |\partial_r v|^2 r^{D-1} dr + C\int_R^\infty \frac{v^2}{r^2} r^{D-1} dr. \quad (2.9)$$

Thus,

$$\begin{aligned} \int_R^\infty |v|^{2^*} r^{D-1} dr &\leq \int_R^\infty v^{2^*-2} r^2 \frac{v^2}{r^2} r^{D-1} dr \\ &\leq C_1\delta^{2^*-2}\int_R^\infty \frac{v^2}{r^2} r^{D-1} dr, \end{aligned}$$

where $C_1 = 2^{2/(D-2)}$. Thus

$$\frac{1}{2}\int_R^\infty |\partial_r v|^2 r^{D-1} dr - \frac{1}{2^*}\int_R^\infty |v|^{2^*} r^{D-1} dr \geq \frac{1}{2}\int_R^\infty |\partial_r v|^2 r^{D-1} dr - \frac{C_1}{2^*}\delta^{2^*-2}\int_R^\infty \frac{v^2}{r^2} r^{D-1} dr.$$

Therefore on re-arranging (2.9) we need to prove that

$$\left(\frac{1}{2}-C\right)\int_R^\infty |\partial_r v|^2 r^{D-1}dr - \frac{C_1}{2^*}\delta^{2^*-2}\int_R^\infty \frac{v^2}{r^2}r^{D-1}dr \geq C\int_R^\infty \frac{v^2}{r^2}r^{D-1}dr.$$

Now we claim that

$$\int_R^\infty \frac{v^2}{r^2}r^{D-1}dr \leq C_3\int_R^\infty |\partial_r v|^2 r^{D-1}dr,$$

where C_3 is the same constant as in the original Hardy's inequality and therefore only depends on the dimension D. We prove this as follows. Let $k \geq 0$ then

$$\begin{aligned}
0 &\leq \int_R^\infty \left(\frac{v}{r}+k\partial_r v\right)^2 r^{D-1}dr \\
&= \int_R^\infty \frac{v^2}{r^2}+k^2|\partial_r v|^2 + 2k\frac{v}{r}\partial_r v r^{D-1}dr \\
&= \int_R^\infty \left(\frac{v^2}{r^2}+k^2|\partial_r v|^2\right) r^{D-1}dr + k\int_R^\infty \frac{\partial_r v^2}{r}r^{D-1}dr \\
&= \int_R^\infty \left(\frac{v^2}{r^2}+k^2|\partial_r v|^2\right) r^{D-1}dr - (D-2)k\int_R^\infty \frac{v^2}{r^2}r^{D-1}dr - v^2(R)kR^{D-2} \\
&= (1-(D-2)k)\int_R^\infty \frac{v^2}{r^2}r^{D-1}dr + k^2\int_R^\infty |\partial_r v|^2 r^{D-1}dr - kv^2(R)R^{D-2}.
\end{aligned}$$

Then

$$f(k) = (1-(D-2)k)\int_R^\infty \frac{v^2}{r^2}r^{D-1}dr + k^2\int_R^\infty |\partial_r v|^2 r^{D-1}dr - kv^2(R)R^{D-2}$$

is polynomial in k and attains its minimal value at

$$k_0 = \frac{v^2(R)R^{D-2} + (D-2)\int_R^\infty \frac{v^2}{r^2}r^{D-1}dr}{2\int_R^\infty |\partial_r v|^2 r^{D-1}dr} = \frac{v^2(R)R^{D-2}}{2\int_R^\infty |\partial_r v|^2 r^{D-1}dr} + \frac{D-2}{2}T \geq \frac{D-2}{2}T.$$

where $T = \frac{\int_R^\infty v^2/r^2 r^{D-1}dr}{\int_R^\infty |\partial_r v|^2 r^{D-1}dr}$. Therefore

$$\begin{aligned}
0 &\leq (1-(D-2)k_0)T + k_0^2 - k_0\frac{v^2(R)R^{D-2}}{\int_R^\infty |\partial_r v|^2 r^{D-1}dr} \\
0 &\leq (1-(D-2)k_0)T + k_0^2 - k_0(2k_0-(D-2)T) \\
k_0^2 &\leq T
\end{aligned}$$

and thus

$$\frac{(D-2)^2}{4}T^2 \leq T \implies T \leq \frac{4}{(D-2)^2}$$

which yields the desired inequality. Thus

$$\begin{aligned}
&\left(\frac{1}{2}-C\right)\int_R^\infty |\partial_r v|^2 r^{D-1}dr - \frac{1}{2^*}\int_R^\infty |v|^{2^*}r^{D-1}dr \\
&\geq \left(\frac{1}{2}-C\right)\int_R^\infty |\partial_r v|^2 r^{D-1}dr - \frac{C_1}{2^*}\delta^{2^*-2}\int_R^\infty \frac{v^2}{r^2}r^{D-1}dr \\
&\geq \left(\frac{C_3^{-1}}{2}-CC_3^{-1}\right)\int_R^\infty \frac{v^2}{r^2}r^{D-1}dr - \frac{C_1}{2^*}\delta^{2^*-2}\int_R^\infty \frac{v^2}{r^2}r^{D-1}dr \\
&= \left(\frac{C_3^{-1}}{2}-CC_3^{-1}-\frac{C_1\delta^{2^*-2}}{2}\right)\int_R^\infty \frac{v^2}{r^2}r^{D-1}dr \\
&\geq C\int_R^\infty \frac{v^2}{r^2}r^{D-1}dr
\end{aligned}$$

provided

$$\delta^{2^*-2} < \frac{C_3^{-1}(1-2C)-2C}{C_1}$$

where $C_3^{-1} = \frac{(D-2)^2}{4}$ and $C_1 = 2^{2/(D-2)}$. Thus for instance if we choose $C = \frac{C_3^{-1}}{4(1+C_3^{-1})}$ then

$$\frac{C_3^{-1}(1-2C)}{C_1} - \frac{2C}{C_1} = \frac{C_3^{-1}}{C_1}\left(1-2C\left(\frac{1+C_3^{-1}}{C_3^{-1}}\right)\right) = \frac{C_3^{-1}}{2C_1}$$

and so we need $\delta^{2^*-2} < \frac{C_3^{-1}}{2C_1}$ to get a constant $C = \frac{C_3^{-1}}{4(1+C_3^{-1})}$ in the desired inequality (2.9). □

Lemma 2.14 (Propagation of small localized $\mathcal{E}$ norm)**.** *There exists $\delta, C > 0$ with the following properties. Let $I \ni 0$ be a time interval and let $u(t) \in \mathcal{E}$ be a solution to* (1.1) *on I with initial data $u(0) = u_0$. Let $0 < r_1 < r_2 < \infty$. Suppose that*

$$\|u_0\|_{\mathcal{E}(r_1/2, 2r_2)} \leq \delta.$$

Then,

$$\|u(t)\|_{\mathcal{E}(r_1, r_2)} \leq C\delta$$

for all $t \in I$ small enough, $t \leq \delta r_1^2$.

Proof. Let $\phi(r)$ be a smooth cut-off function such that $\phi \equiv 1$ on $[r_1, r_2]$ and $\phi \equiv 0$ on $(0, r_1/2] \cup [2r_2, \infty)$. Then $\partial_t \phi = 0$ and thus by (2.7) we have

$$\|u(t)\|^2_{\mathcal{E}(r_1,r_2)} \lesssim \|u_0\|^2_{\mathcal{E}(r_1/2, 2r_2)} + \frac{t}{r_1^2} + \frac{\sqrt{t}}{r_1}.$$

Then for short times satisfying $t < r_1^2$ we get the desired estimate. □

Lemma 2.15 (Short time evolution close to W)**.** *Let $\iota \in \{-1, 1\}$. There exists $\delta_0 > 0$ and a function $\epsilon_0 : [0, \delta_0] \to [0, \infty)$ with $\epsilon_0(\delta) \to 0$ as $\delta \to 0$ with the following properties. Let $v_0 \in \mathcal{E}$ and let $v(t)$ denote the unique solution to* (1.1) *with $v_0(0) = v_0$. Let $\mu_0, T_0 > 0$ and suppose that*

$$\|v_0 - \iota W_{\mu_0}\|_{\mathcal{E}} + \frac{T_0^2}{\mu_0} = \delta \leq \delta_0$$

Then, $T_0 < T_+(v_0)$ and

$$\sup_{t\in[0,T_0]} \|v(t) - \iota W_{\mu_0}\|_{\mathcal{E}} < \epsilon_0(\delta)$$

Proof. By rescaling we may assume $\mu_0 = 1$. The result is then a particular case of the local Cauchy theory, in particular the continuity of the data to solution map at W. □

Lemma 2.16 (Localized short time evolution close to W)**.** *Let $\iota \in \{-1,1\}$. There exists $\delta_0 > 0$ and a function $\epsilon_0 : [0,\delta_0] \to [0,\infty)$ with $\epsilon_0(\delta) \to 0$ as $\delta \to 0$ with the following properties. Let $u_0 \in \mathcal{E}, T_0 < T_+(u_0)$, and let $u(t)$ denote the unique solution to (1.1) with $u_0(0) = u_0$. Let $\mu_0 > 0, 0 < r_1 < r_2 < \infty$ and suppose that*

$$\|u_0 - \iota W_{\mu_0}\|_{\mathcal{E}(r_1/2,2r_2)} + \frac{T_0^2}{\mu_0} = \delta \leq \delta_0$$

Then,

$$\|u(t) - \iota W_{\mu_0}\|_{\mathcal{E}(r_1,r_2)} < \epsilon_0(\delta)$$

for all $0 < t \leq T_0$ such that $T_0 \leq \delta r_1^2$.

Proof. Consider $v_0 := \phi u_0 + (1-\phi)\iota W_{\mu_0}$, where ϕ is the same function defined in Lemma 2.14. Then

$$\|v_0 - \iota W_{\mu_0}\|_{\mathcal{E}} = \|\phi u_0 + (1-\phi)\iota W_{\mu_0} - \iota W_{\mu_0}\|_{\mathcal{E}} = \|\phi\,(u_0 - \iota W_{\mu_0})\|_{\mathcal{E}} \leq \|u_0 - \iota W_{\mu_0}\|_{\mathcal{E}(r_1/2,2r_2)}.$$

By taking δ_0 sufficiently small we see that v_0, μ_0, T_0 satisfy the hypothesis of Lemma 2.15. □

Lemma 2.17. *If $\iota_n \in \{-1,0,1\}, 0 < t_n^2 < r_n \ll \mu_n \ll R_n$ and u_n a sequence of solutions of (1.1) such that $u_n(t)$ is defined for $t \in [0,t_n]$ and*

$$\lim_{n\to\infty} \|u_n(0) - \iota_n W_{\mu_n}\|_{\mathcal{E}(r_n/2,2R_n)} = 0,$$

then

$$\lim_{n\to\infty} \sup_{t\in[0,t_n]} \|u_n(t) - \iota_n W_{\mu_n}\|_{\mathcal{E}(r_n,R_n)} = 0.$$

Proof. This is a direct consequence of Lemma (2.14) when $\iota_n = 0$ and Lemma (2.16) when $\iota_n \in \{-1,1\}$. □

3. Sequential bubbling for finite time blow-up solutions

Lemma 3.1 (Identification of the body map)**.** *Let $u_0 \in \mathcal{E}$ and let $u(t)$ be the solution to (1.1). Suppose that $T_+(u_0) < \infty$ and let $I_* = [0,T_+)$. There exist a mapping $u^* \in \mathcal{E}$ such that for any $r_0 > 0$,*

$$\lim_{t\to T_*} \|u(t) - u^*\|_{\mathcal{E}(r\geq r_0)} = 0. \tag{3.1}$$

Moreover, there exists $L > 0$ such that for each $r_0 \in (0,\infty]$,

$$\lim_{t\to T_+} E\left(u(t);0,r_0\right) = L + E\left(u^*;0,r_0\right), \tag{3.2}$$

and in particular, $\lim_{r_0\to 0}\lim_{t\to T_+} E\left(u(t);0,r_0\right) = L$.

Remark 3.2. See the works of Struwe [Str85] and Qing [Qin95] for analogous results for the harmonic maps.

Lemma 3.1 can be established by essentially following the proof of the main theorem established in [Du13]. In particular, by the arguments in Section 3 of [Du13] (applied to the domain $\Omega = \mathbb{R}^n$) we have

Theorem 3.3. *Let u be a solution of the following problem:*

$$\begin{cases} u_t = \Delta u + |u|^{p-1}u, & \text{in } \mathbb{R}^n \times (0, T_+), \\ u(0) \text{ given}, \end{cases} \tag{3.3}$$

for $n \geq 3$ and $p = \frac{n+2}{n-2}$. Assume that

$$\limsup_{t \to T_+} \|u(t)\|_{\mathcal{E}} < +\infty$$

and that there is no sub-convergence for $u(t)$ strongly in $H^1_{loc}(\mathbb{R}^n)$ as $t \to T_+$. Then there exist $\{t_k\}, t_k \to T_+$, and $u^* \in H^1_{loc}(\mathbb{R}^n)$, a steady state of (3.3), such that $u(t_k) \rightharpoonup u^*$ weakly in $H^1_{loc}(\mathbb{R}^n)$. Furthermore, there exist $x^1, \ldots, x^N \in \mathbb{R}^n, x^j_{i,k} \to x^j, \lambda^j_{i,k} \to 0$ as $k \to \infty$ for $i = 1, 2, \ldots, I_j$ and $j = 1, \ldots, N$, such that the following hold:

$$u(t_k) - \sum_{j=1}^{N} \sum_{i=1}^{I_j} \left(\lambda^j_{i,k}\right)^{\frac{-2}{p-1}} W\left(\frac{x - x^j_{i,k}}{\lambda^j_{i,k}}\right) \to u^* \quad \text{in } H^1_{loc}(\mathbb{R}^n), \quad \text{as } k \to \infty,$$

and the energy identity holds for each $r_0 > 0$

$$\lim_{k \to \infty} E\left(u(t_k); 0, r_0\right) = E\left(u^*; 0, r_0\right) + L \tag{3.4}$$

where $L = \left(\sum_{j=1}^{N} I_j\right) E(W)$ and W is a bubble solving the elliptic equation

$$\Delta W + |W|^{p-1}W = 0, \text{ on } \mathbb{R}^n.$$

Remark 3.4. Note that the bubbles in the above decomposition are not necessarily the standard Aubin Talenti bubbles since the elliptic PDE satisfied by the bubbles also admits sign-changing solutions.

Proof of Lemma 3.1. To see (3.1) observe that the singularity in the radial case can only occur at the origin $r = 0$ since otherwise, the solution will blow up at points lying on a sphere which is an uncountable set. Since energy is quantized, the energy identity would imply that the solution itself has infinite energy which violates the finite energy assumption. Furthermore, from the bubble tree convergence in Theorem 3.3 we see that the solution converges smoothly to the radiation/body-map u^* outside the singular set defined as

$$\mathcal{S} = \left\{ x \in \mathbb{R}^n : \lim_{R \to 0} \liminf_{t_k \to T_+} \int_{B_R(x)} |u(t_k)|^{p+1} \geq \varepsilon_0 \right\},$$

and therefore (3.1) holds. Furthermore, (3.2) also follows from the H^1_{loc} convergence in Theorem 3.3, which in particular implies that $\lim_{t \to T_+} \int |\nabla u(\cdot, t)|^2$ exists. □

Proposition 3.5 (Sequential bubbling for solutions that blow up in finite time)**.** *Let $u_0 \in \mathcal{E}$, and let $u(t)$ denote the solution to (1.1) with initial data u_0. Suppose that $T_+(u_0) < \infty$. There exist a mapping $u^* \in \mathcal{E}$, an integer $N \geq 1$, a sequence of times $t_n \to T_+$, signs $\vec{\iota} \in \{-1, 1\}^N$, a sequence of scales $\vec{\lambda}_n \in (0, \infty)^N$, and an error g_n defined by*

$$u(t_n) = \sum_{j=1}^{N} \iota_j \mathcal{W}_{\lambda_n} + u^* + g_n,$$

with the following properties:

(i) The integer $N \geq 1$ and the body map u^* satisfy,

$$\lim_{t \to T_+} E(u(t)) = NE(W) + E(u^*); \tag{3.5}$$

(ii) for any $0 < \alpha < A$

$$\lim_{t \to T_+} E(u(t); 0, \alpha(T_+ - t)^{\frac{1}{2}}) = NE(W), \tag{3.6}$$

$$\lim_{t \to T_+} E(u(t); \alpha(T_+ - t)^{\frac{1}{2}}, A(T_+ - t)^{\frac{1}{2}}) = 0, \tag{3.7}$$

$$\lim_{t \to T_+} E(u(t) - u^*; \alpha(T_+ - t)^{\frac{1}{2}}, \infty) = 0, \tag{3.8}$$

and there exists $0 < T_0 < T_+$ *and function* $\rho : [T_0, T_+) \to (0, \infty)$ *satisfying,*

$$\lim_{t \to T_*} \big((\rho(t)/\sqrt{T_+ - t}) + \|u(t) - u^*\|_{\mathcal{E}(\rho(t))}\big) = 0; \tag{3.9}$$

(iii) the error g_n *and the scales* $\vec{\lambda}_n$ *satisfy,*

$$\lim_{n \to \infty} \Big(\|g_n\|_{\mathcal{E}}^2 + \sum_{j=1}^{N} \Big(\frac{\lambda_{n,j}}{\lambda_{n,j+1}}\Big)^{\frac{D-2}{2}}\Big)^{\frac{1}{2}} = 0,$$

where here we adopt the convention that $\lambda_{n,N+1} := (T_+ - t_n)^{\frac{1}{2}}$.

Proof of Proposition 3.5. Let $u(t) \in \mathcal{E}$ be a solution to (1.1) blowing up at time $T_+ > 0$. By (2.1) we can find a sequence $t_n \to T_+$ so that,

$$(T_+ - t_n)^{\frac{1}{2}} \|\mathcal{T}(u(t_n))\|_{L^2} \to 0 \quad \text{as} \quad n \to \infty.$$

Applying Lemma 1.6 with $\rho_n := (T_+ - t_n)^{\frac{1}{2}}$, implies that there exist $N \geq 0$, $m_0 \in \mathbb{Z}$, $\vec{\iota} \in \{-1, 1\}^N$, $\vec{\lambda}_n \in (0, \infty)^N$ such that after passing to a subsequence, we have

$$\lim_{n \to \infty} \Big(\|u(t_n) - \mathcal{W}(\vec{\iota}, \vec{\lambda}_n)\|_{\mathcal{E}(r \leq A(T_+ - t_n)^{\frac{1}{2}})}^2 + \sum_{j=1}^{N-1} \Big(\frac{\lambda_{n,j}}{\lambda_{n,j+1}}\Big)^{\frac{D-2}{2}}\Big) = 0 \tag{3.10}$$

for each $A > 0$, and $\lambda_{n,N} \lesssim (T_+ - t_n)^{\frac{1}{2}}$. Consider two smooth cutoff functions

$$\phi \equiv 1 \text{ on } [r_1, r_2], \quad \phi \equiv 0 \text{ on } (0, r_1/2] \cap [2r_2, \infty)$$
$$\chi \equiv 1 \text{ on } (0, r_2], \quad \chi \equiv 0 \text{ on } [2r_2, \infty)$$

where we will choose positive parameters r_1 and r_2 appropriately. Define the localized $\mathcal{E}$-norm on the annulus

$$\tilde{\Theta}_{[r_1, r_2]}(t) = \int \phi(r)^2 \tilde{e}(u(t, r)) r^{D-1} dr, \quad \tilde{\Theta}^*_{[r_1, r_2]} = \int \phi(r)^2 \tilde{e}(u^*(r)) r^{D-1} dr.$$

From (2.6) we see that for each $0 < s < \tau < T_+$ we have,

$$\begin{aligned}
&\left|\tilde{\Theta}_{[r_1,r_2]}(\tau) - \tilde{\Theta}_{[r_1,r_2]}(s)\right| \\
&\lesssim \int_s^\tau \|\partial_t u\|_{L^2}^2 dt + \left(\int_{\mathbb{R}_+}\int_s^\tau |u|^{2p}\phi^2\right)^{1/2}\left(\int_{\mathbb{R}_+}\int_s^\tau (\partial_t u)^2\phi^2\right)^{1/2} \\
&\quad + \left(\int_{\mathbb{R}_+}\int_s^\tau (\partial_t u)^2\phi^2(\partial_r\phi)^2\right)^{1/2}\left(\int_{\mathbb{R}_+}\int_s^\tau (\partial_r u)^2\right)^{1/2} + \left(\int_s^\tau\int_{\mathbb{R}_+}\frac{|u|^2}{r^4}\phi^2\right)^{1/2}\left(\int_s^\tau\int_{\mathbb{R}_+}(\partial_t u)^2\phi^2\right)^{1/2} \\
&\lesssim \int_s^{T_+} \|\partial_t u(t)\|_{L^2}^2\,\mathrm{d}t + \sqrt{\int_s^\tau\int_{r_1}^{r_2}\frac{1}{r^3}drdt}\sqrt{\int_s^\tau \|\partial_t u\|_{L^2}^2 dt} \\
&\quad + \frac{(\tau-s)^{1/2}}{r_1}\sqrt{\int_s^\tau \|\partial_t u\|_{L^2}^2 dt}\left(\int_s^\tau\int_{r_1}^{r_2}\frac{1}{r^3}dr\right)^{1/2}\sqrt{\int_s^\tau \|\partial_t u\|_{L^2}^2 dt} \\
&\lesssim \int_s^{T_+} \|\partial_t u(t)\|_{L^2}^2\,\mathrm{d}t + \frac{(T_+-s)^{1/2}}{r_1}\sqrt{\int_s^{T_+} \|\partial_t u\|_{L^2}^2 dt}
\end{aligned}$$

Let $s \to T_+$ we see that $\lim_{s\to T_+}\tilde{\Theta}_{[r_1,r_2]}(s)$ exists. We first prove

$$\lim_{t\to T_+} \widetilde{E}(u(t);\alpha(T_+-t)^{1/2},r_0) = \widetilde{E}(u^*;0,r_0) \tag{3.11}$$

for any $r_0 \in (0,+\infty]$. To see this observe that set $r_2 = r_0$ and let $0 < r' < \frac{r_1}{2} < r_1$, then

$$\tilde{\Theta}_{[r_1,r_2]}(\tau) - \tilde{\Theta}^*_{[r_1,r_2]} = \int_{r'}^{2r_2} \phi(r)^2\left(\tilde{e}(u) - \tilde{e}(u^*)\right).$$

As $\tau \to T_+$ the expression on the RHS tends to zero by (3.1). Thus choosing $r_1 = \alpha(T_+-s)^{1/2}$ for any $\alpha > 0$ we have

$$\left|\tilde{\Theta}^*_{[\alpha(T_+-s)^{1/2},r_0]} - \tilde{\Theta}_{[\alpha(T_+-s)^{1/2},r_0]}(s)\right| \lesssim \int_s^{T_+} \|\partial_t u(t)\|_{L^2}^2\,\mathrm{d}t + \frac{1}{\alpha}\sqrt{\int_s^{T_+} \|\partial_t u\|_{L^2}^2 dt}$$

which implies that as $s \to T_+$ we have

$$\lim_{s\to T_+} \tilde{\Theta}_{[\alpha(T_+-s)^{1/2},r_0]}(s) = \tilde{\Theta}^*_{[\alpha(T_+-s)^{1/2},r_0]}.$$

Thus, in particular, we have

$$\tilde{\Theta}_{[2\alpha(T_+-t)^{1/2},r_0/2]}(s) \le \widetilde{E}[u(t);\alpha(T_+-s)^{1/2},r_0] \le \tilde{\Theta}_{[\alpha(T_+-t)^{1/2},r_0]}(s)$$

which on taking limits as $t \to T_+$ implies that

$$\lim_{t\to T_+} \widetilde{E}(u(t);\alpha(T_+-t)^{1/2},r_0) = \widetilde{E}(u^*;0,r_0).$$

Proof of (3.8): We will show that for given $\epsilon > 0$ there exists $s_0 > 0$ such that for all $\tau \in [s_0, T_+)$ we have

$$\widetilde{E}(u(\tau)-u^*;\alpha_1(T_+-\tau)^{\frac{1}{2}},\infty) \lesssim \epsilon.$$

To this end note the following estimate

$$\begin{aligned}
\widetilde{E}(u(\tau)-u^*;&\alpha_1(T_+-\tau)^{\frac{1}{2}},\infty) \le \widetilde{E}(u(\tau)-u^*;\alpha_1(T_+-\tau)^{\frac{1}{2}},r_0) + \widetilde{E}(u(\tau)-u^*;r_0,\infty) \\
&\le 2\widetilde{E}(u(\tau);\alpha_1(T_+-\tau)^{\frac{1}{2}},r_0) + 2\widetilde{E}(u^*;\alpha_1(T_+-\tau)^{\frac{1}{2}},r_0) + \widetilde{E}(u(\tau)-u^*;r_0,\infty).
\end{aligned}$$

For τ sufficiently close to T_+ we can first choose $r_0 > 0$ small enough such that

$$\widetilde{E}(u^*; \alpha_1(T_+ - \tau)^{1/2}, r_0) \le \widetilde{E}(u^*; 0, r_0) \le \epsilon.$$

Next using (3.1) we see that for τ sufficiently close to T_+

$$\widetilde{E}(u(\tau) - u^*; r_0, \infty) \le \epsilon.$$

Finally from (3.11) we have that

$$\widetilde{E}(u(\tau); \alpha_1(T_+ - \tau)^{1/2}, r_0) \le 2\epsilon.$$

Thus combining all the above estimates we have

$$\widetilde{E}(u(\tau) - u^*; \alpha_1(T_+ - \tau)^{\frac{1}{2}}, \infty) \le 4\epsilon$$

which establishes (3.8). Furthermore this also establishes (3.9) i.e.,

$$\lim_{t \to T_+} \|u(t) - u^*\|_{\mathcal{E}(r \ge \alpha(T_+ - t)^{\frac{1}{2}})} = 0,$$

Proof of (3.7): Let $0 < \alpha < A < \infty$ then we first show that

$$\lim_{s \to T_+} \widetilde{E}(u(s); \alpha(T_+ - s)^{\frac{1}{2}}, A(T_+ - s)^{\frac{1}{2}}) = 0$$

To see this we just make use of (3.8). Since

$$\begin{aligned} 0 \le & \widetilde{E}(u(s); \alpha(T_+ - s)^{\frac{1}{2}}, A(T_+ - s)^{\frac{1}{2}}) = \widetilde{E}(u(s) - u^* + u^*; \alpha(T_+ - s)^{\frac{1}{2}}, A(T_+ - s)^{\frac{1}{2}}) \\ & \le 2\widetilde{E}(u(s) - u^*; \alpha(T_+ - s)^{\frac{1}{2}}, A(T_+ - s)^{\frac{1}{2}}) + 2\widetilde{E}(u^*; \alpha(T_+ - s)^{\frac{1}{2}}, A(T_+ - s)^{\frac{1}{2}}) \\ & \le 2\widetilde{E}(u(s) - u^*; \alpha(T_+ - s)^{\frac{1}{2}}, \infty) + 2\widetilde{E}(u^*; \alpha(T_+ - s)^{\frac{1}{2}}, A(T_+ - s)^{\frac{1}{2}}) \end{aligned}$$

which tends to zero as $s \to T_+$. Thus

$$\lim_{s \to T_+} \widetilde{E}(u(s); \alpha(T_+ - s)^{\frac{1}{2}}, A(T_+ - s)^{\frac{1}{2}}) = 0.$$

By the same argument if we define the energy density (1.2) with just the gradient term, i.e.

$$\bar{E} = \int (\partial_r u)^2 r^{D-1} dr$$

then we can also deduce that

$$\lim_{s \to T_+} \bar{E}(u(s); \alpha(T_+ - s)^{\frac{1}{2}}, A(T_+ - s)^{\frac{1}{2}}) = 0.$$

Consequently, if we consider the nonlinear energy then

$$E(u(t); \alpha(T_+ - t)^{1/2}, A(T_+ - t)^{1/2}) = \frac{1}{2}\bar{E}(u(t); \alpha(T_+ - t)^{1/2}, A(T_+ - t)^{1/2}) - \frac{1}{2^*}\int_{\alpha(T_+ - t)^{1/2}}^{A(T_+ - t)^{1/2}} |u(t)|^{2^*}.$$

The first term in the RHS of the above equality tends to zero so we only need to show that the second term also tends to zero. For this, we observe that by using a cutoff function and then Sobolev inequality we get

$$0 \le \int_{\alpha(T_+ - t)^{1/2}}^{A(T_+ - t)^{1/2}} |u|^{2^*} r^{D-1} dr \lesssim \widetilde{E}\left(u; \frac{1}{2}\alpha(T_+ - t)^{1/2}, 2A(T_+ - t)^{1/2}\right)^{2^*} \to 0$$

as $t \to T_+$ and therefore we get

$$\lim_{t \to T_+} E(u(t); \alpha(T_+ - t)^{\frac{1}{2}}, A(T_+ - t)^{\frac{1}{2}}) = 0.$$

Proof of (3.5) and (3.6). From the decomposition (3.10) we deduce that

$$\frac{\lambda_{n,N}}{(T_+ - t_n)^{\frac{1}{2}}} \to 0 \text{ as } n \to \infty,$$

which from (3.4) implies that $L = NE(W)$. Then

$$\begin{aligned}
\lim_{t\to T_+} E(u(t)) &= L + E(u^*) \\
&= NE(W) + E(u^*) \\
&= \lim_{t\to T_+} E(u(t); 0, \alpha(T_+ - t)^{1/2}) + \lim_{t\to T_+} E(u(t); \alpha(T_+ - t)^{1/2}, +\infty) \\
&= \lim_{t\to T_+} E(u(t); 0, \alpha(T_+ - t)^{1/2}) + \lim_{t\to T_+} E(u(t) - u^* + u^*; \alpha(T_+ - t)^{1/2}, +\infty) \\
&= \lim_{t\to T_+} E(u(t); 0, \alpha(T_+ - t)^{1/2}) + \lim_{t\to T_+} E(u(t) - u^*; \alpha(T_+ - t)^{1/2}, +\infty) \\
&\quad + \lim_{t\to T_+} E(u^*; \alpha(T_+ - t)^{1/2}, +\infty) + \mathcal{O}\left(\lim_{t\to T_+} \|u(t) - u^*\|_{\mathcal{E}(\alpha(T_+-t)^{1/2}, +\infty)}\right) \\
&= \lim_{t\to T_+} E(u(t); 0, \alpha(T_+ - t)^{1/2}) + E(u^*).
\end{aligned}$$

This proves

$$\lim_{t\to T_+} E(u(t)) = NE(W) + E(u^*)$$

$$\lim_{t\to T_+} E(u(t); 0, \alpha(T_+ - t)^{\frac{1}{2}}) = NE(W)$$

as desired. □

4. Sequential bubbling for global solutions

Proposition 4.1 (Sequential bubbling for global-in-time solutions)**.** *Let $u_0 \in \mathcal{E}$ and let $u(t)$ denote the solution to* (1.1) *with initial data u_0. Suppose that $T_+(u_0) = \infty$. Then there exist $T_0 > 0$, an integer $N \geq 0$, a sequence of times $t_n \to \infty$, signs $\vec{\iota} \in \{-1, 1\}^N$, a sequence of scales $\vec{\lambda}_n \in (0, \infty)^N$, and an error g_n defined by*

$$u(t_n) = \sum_{j=1}^{N} \iota_j Q_{\lambda_n} + g_n$$

with the following properties:

(i) the integer $N \geq 0$ satisfies,

$$\lim_{t\to\infty} E(u(t)) = NE(W); \tag{4.1}$$

(ii) for every $\alpha > 0$,

$$\lim_{t\to\infty} E(u(t); \alpha\sqrt{t}, \infty) = 0, \tag{4.2}$$

and there exists $T_0 > 0$ and a function $\rho : [T_0, \infty) \to (0, \infty)$ such that

$$\lim_{t\to\infty} \left(\frac{\rho(t)}{\sqrt{t}} + \|u(t)\|_{\mathcal{E}(r\geq\rho(t))}\right) = 0; \tag{4.3}$$

(iii) the scales $\vec{\lambda}_n$ and the sequence g_n satisfy,

$$\lim_{n\to\infty}\Big(\|g_n\|_{\mathcal{E}}^2+\sum_{j=1}^{N}\Big(\frac{\lambda_{n,j}}{\lambda_{n,j+1}}\Big)^{\frac{D-2}{2}}\Big)^{\frac12}=0 \tag{4.4}$$

where here we adopt the convention that $\lambda_{n,j+1}:=t_n^{\frac12}$.

Proof. Let $u(t)\in\mathcal{E}$ be a heat flow defined globally in time. By (2.1) we can find a sequence $t_n\to\infty$ so that, $t_n^{\frac12}\|\mathcal{T}(u(t_n))\|_{L^2}\to 0$ as $n\to\infty$. We can now apply Proposition 1.6 which yields $N\geq 0$, $\vec{\iota}\in\{-1,1\}^N$, $\vec{\lambda}_n\in(0,\infty)^N$ such that after passing to a subsequence, we have

$$\lim_{n\to\infty}\Big(\|u(t_n)-\mathcal{W}(\vec{\iota},\vec{\lambda}_n)\|^2_{\mathcal{E}(r\leq A\sqrt{t_n})}+\sum_{j=1}^{N-1}\Big(\frac{\lambda_{n,j}}{\lambda_{n,j+1}}\Big)^{\frac{D-2}{2}}\Big)=0 \tag{4.5}$$

for each $A>0$, and moreover that $\lambda_{n,N}\lesssim t_n^{\frac12}$. Fix $\alpha>0$ and let $\varepsilon>0$ be small enough such that by (2.1) and the fact that $\|u_0\|_{\mathcal{E}}<+\infty$ we can find $T_0=T_0(\varepsilon)>0$ such that,

$$\frac{4\|u_0\|_{\mathcal{E}}}{\alpha}\Big(\int_{T_0}^{\infty}\int_0^{\infty}(\partial_t u(t,r))^2\,r\,\mathrm{d}r\,\mathrm{d}t\Big)^{\frac12}\leq\varepsilon. \tag{4.6}$$

Next, choose $T_1\geq T_0$ so that

$$\|u(T_0)\|_{\mathcal{E}(\alpha\sqrt{T}/4,\infty)}\leq\varepsilon \tag{4.7}$$

Fixing any such T, we set

$$\phi(t,r)=\phi_T(r)=1-\chi(4r/\alpha\sqrt{T})\quad\text{for}\quad t\in[T_0,T]$$

where $\chi(r)$ is a smooth function on $(0,\infty)$ such that $\chi(r)=1$ for $r\leq 1$, $\chi(r)=0$ if $r\geq 4$, and $|\chi'(r)|\leq 1$ for all $r\in(0,\infty)$. Since $\frac{\mathrm{d}}{\mathrm{d}t}\phi(t,r)=0$ for $t\in[T_0,T]$ it follows from (2.8) that

$$\begin{aligned}
&\int_{\mathbb{R}_+}\tilde{\mathbf{e}}(u(T))\phi^2-\int_{\mathbb{R}_+}\tilde{\mathbf{e}}(u(T_0))\phi^2\\
&\leq -2\int_{\mathbb{R}_+}\int_{T_0}^{T}(\partial_t u)^2\phi^2+2\Big(\int_{\mathbb{R}_+}\int_{T_0}^{T}|u|^{2p}\phi^2\Big)^{1/2}\Big(\int_{\mathbb{R}_+}\int_{T_0}^{T}(\partial_t u)^2\phi^2\Big)^{1/2}\\
&\quad+4\Big(\int_{\mathbb{R}_+}\int_{T_0}^{T}(\partial_t u)^2\phi^2(\partial_r\phi)^2\Big)^{1/2}\Big(\int_{\mathbb{R}_+}\int_{T_0}^{T}(\partial_r u)^2\Big)^{1/2}+2\Big(\int_s^{\tau}\int_{\mathbb{R}_+}\frac{|u|^2}{r^4}\phi^2\Big)^{1/2}\Big(\int_s^{\tau}\int_{\mathbb{R}_+}(\partial_t u)^2\phi^2\Big)^{1/2}\\
&\lesssim\Big(\int_{T_0}^{T}\int_{\frac{\alpha\sqrt{T}}{4}}^{\infty}\frac{1}{r^3}drdt\Big)^{1/2}\Big(\int_{T_0}^{T}\int_0^{\infty}(\partial_t u)^2\Big)^{1/2}+\Big(\frac{T-T_0}{T}\Big)^{1/2}\Big(\int_{\mathbb{R}_+}\int_{T_0}^{T}(\partial_t u)^2\phi^2(\partial_r\phi)^2\Big)^{1/2}\\
&\lesssim\Big(\int_{T_0}^{\infty}\int_0^{\infty}(\partial_t u)^2\Big)^{1/2}
\end{aligned}$$

where the constant in the above inequality depends on $\sup_{t\in[0,\infty)}\|u(t)\|_{\mathcal{E}}$ and α. So in particular we can make this small by choosing T_0 large enough such that $\tilde{C}\left(\int_{T_0}^{\infty}\int_0^{\infty}(\partial_t u)^2\right)^{1/2}\leq\frac{\varepsilon}{2}$. Using the above together with (4.6) and (4.7) we find that the Dirichlet energy can be made arbitrarily small

$$\widetilde{E}(u(T);\alpha\sqrt{T},\infty)\leq\varepsilon.$$

for all $T \geq T_1$ and $t \in [T_0, T]$. Thus by the localized coercivity lemma 2.13 we see that that $E(u(T); \alpha\sqrt{T}, \infty) \geq 0$ and moreover since

$$E(u(T); \alpha\sqrt{T}, \infty) \lesssim \widetilde{E}(u(T); \alpha\sqrt{T}, \infty) \leq \varepsilon$$

we get (4.2). Since we proved (4.2) for any $\alpha > 0$ there exists a curve ρ such that (4.3) holds. Returning to the sequential decomposition we see from (4.5), the fact that $\lambda_{n,N} \lesssim t_n^{\frac{1}{2}}$, and from (4.2) that we must have

$$\lim_{n\to\infty} \frac{\lambda_{n,N}}{t_n^{\frac{1}{2}}} = 0.$$

Then, (4.4) follows from the above, (4.3) and (4.5). Moreover, we see that $\lim_{n\to\infty} E(u(t_n)) = NE(W)$ and the continuous limit (4.1) then follows from the fact that $E(u(t))$ is non-increasing. □

5. Decomposition of the solution and collision intervals

For the remainder of the paper, we fix a solution $u(t) \in \mathcal{E}$ of (1.1), defined on the time interval $I_* = [0, T_*)$ where $T_* < \infty$ or $T_* = +\infty$. When $T_* = +\infty$ we simply set $u^* = 0$. By Proposition 3.5 and 4.1 there exists an integer $N \geq 0$ and a sequence of times $t_n \to T_*$ so that $u(t_n) - u^*$ approaches an N-bubble as $n \to \infty$. We define a localized distance to an N-bubble.

Definition 5.1 (Proximity to a multi-bubble)**.** For all $t \in I$, $\rho \in (0, \infty)$, and $K \in \{0, 1, \ldots, N\}$, we define the *localized multi-bubble proximity function* as

$$\mathbf{d}_K(t; \rho) := \inf_{\vec{\iota}, \vec{\lambda}} \left(\|u(t) - u^* - \mathcal{W}(\vec{\iota}, \vec{\lambda})\|^2_{\mathcal{E}(\rho,\infty)} + \sum_{j=K}^{N} \left(\frac{\lambda_j}{\lambda_{j+1}} \right)^{\frac{D-2}{2}} \right)^{\frac{1}{2}},$$

where $\vec{\iota} := (\iota_{K+1}, \ldots, \iota_N) \in \{-1, 1\}^{D-K}$, $\vec{\lambda} := (\lambda_{K+1}, \ldots, \lambda_N) \in (0, \infty)^{N-K}$, $\lambda_K := \rho$ and $\lambda_{N+1} := \sqrt{t}$.
The *multi-bubble proximity function* is defined by $\mathbf{d}(t) := \mathbf{d}_0(t; 0)$.

Remark 5.2. Observe that $\mathbf{d}_K(t; \rho)$ is small, implies that $u(t) - u^*$ is close to $N - K$ bubbles in the exterior region $r \in (\rho, \infty)$.

From Definition 5.1 and Propositions 3.5 and 4.1 we deduce that there exists a monotone sequence $t_n \to T_*$ such that

$$\lim_{n\to\infty} \mathbf{d}(t_n) = 0. \tag{5.1}$$

Furthermore, proving Theorem 1.1 would follow from showing that

$$\lim_{t\to T_*} \mathbf{d}(t) = 0. \tag{5.2}$$

Note that it suffices to prove Theorem 1.1 in the case when $N \geq 1$ since when $N = 0$, the result follows from (3.5) and (4.1). Furthermore, as a direct consequence of (3.9) and (4.3) we have the following lemma.

Lemma 5.3. *There exists $T_0 > 0$ and function $\rho_N : [T_0, T_*) \to (0, \infty)$ such that*

$$\lim_{t\to T_*} \mathbf{d}_N(t; \rho_N(t)) = 0. \tag{5.3}$$

5.1. **Collision intervals.** To prove Theorem 1.1 we analyze collision intervals. We recall their definition from [JL23a].

Definition 5.4 (Collision interval)**.** Let $K \in \{0, 1, \dots, N\}$. A compact time interval $[a, b] \subset I_*$ is a *collision interval* with parameters $0 < \varepsilon < \eta$ and $N - K$ exterior bubbles if

- $\mathbf{d}(a) \le \varepsilon$ and $\mathbf{d}(b) \ge \eta$,
- there exists a function $\rho_K : [a, b] \to (0, \infty)$ such that $\mathbf{d}_K(t; \rho_K(t)) \le \varepsilon$ for all $t \in [a, b]$.

In this case, we write $[a, b] \in \mathcal{C}_K(\varepsilon, \eta)$.

Definition 5.5 (Choice of K)**.** We define K as the *smallest* nonnegative integer having the following property. There exist $\eta > 0$, a decreasing sequence $\varepsilon_n \to 0$, and sequences $(a_n), (b_n)$ such that $[a_n, b_n] \in \mathcal{C}_K(\varepsilon_n, \eta)$ for all $n \in \{1, 2, \dots\}$.

Next, we observe that $K \ge 1$ since if (5.2) is false then at least one bubble must lose its shape.

Lemma 5.6 (Existence of $K \ge 1$)**.** *If* (5.2) *is false, then K is well defined and $K \in \{1, \dots, N\}$.*

Proof of Lemma 5.6. Assume (5.2) is false, then there exist $\eta > 0$ and a monotone sequence $b_n \to T_*$ such that for all n,

$$\mathbf{d}(b_n) \ge \eta.$$

Next, we will find two sequences (ε_n) and (a_n) such that $[a_n, b_n] \in \mathcal{C}_N(\varepsilon_n, \eta)$. First, using (5.1) we see that there exists $\varepsilon_n \to 0$ and $a_n \le b_n$ such that $\mathbf{d}(a_n) \le \varepsilon_n$. Note that $a_n \to T_*$ and $b_n \to T_*$. To verify the second point in the Definition 5.4 we need to find a curve ρ_N. To this end, we make use of Lemma 5.3 which yields the existence of a curve ρ_N which we restrict to the time interval $[a_n, b_n]$. Then (5.3) yields

$$\lim_{n\to\infty} \sup_{t\in[a_n,b_n]} \mathbf{d}_N(t; \rho_N(t)) = 0.$$

Thus up to changing ε_n, we see that all the conditions of Definition 5.4 are satisfied and hence $[a_n, b_n] \in \mathcal{C}_N(\varepsilon_n, \eta)$. In particular, $K \le N$.

We now show that $K \ge 1$. Suppose to the contrary that $K = 0$. Then by the Definition 5.4 of a collision interval, there exist $\eta > 0$, a sequence of times (c_n) and scales $\rho_n \ge 0$ such that $\mathbf{d}_0(c_n; \rho_n) \le \epsilon_n$ and $\mathbf{d}(c_n) \ge \eta$. Without loss of generality assume that $\eta \le \mathbf{d}(c_n) \le 2\eta$ for each n and η is small enough such that Lemma 2.12 is valid. Then by (3.5) and (4.1) we know that

$$E(u) = NE(W) + E(u^*). \tag{5.4}$$

Whereas since $\mathbf{d}_0(c_n, \rho_n) \le \epsilon_n$, there exist parameters $\rho_n \ll \lambda_{n,1} \ll \cdots \ll \lambda_{n,N} \ll \rho(c_n) \ll c_n$ and signs $\vec{\iota}_n$ such that

$$\left\| u(c_n) - u^*(c_n) - \mathcal{W}\left(\vec{\iota}_n, \vec{\lambda}_n\right) \right\|^2_{\mathcal{E}(\rho_n,\infty)} + \sum_{j=0}^{N} \left(\frac{\lambda_{n,j}}{\lambda_{n,j+1}} \right)^{\frac{D-2}{2}} \lesssim \epsilon_n^2. \tag{5.5}$$

Then since using Lemma 2.4 and the asymptotic orthogonality of the scales we get

$$E(u(c_n); \rho_n, \infty) = NE(W) + E(u^*) + o_n(1) \text{ as } n \to \infty$$

Combing the above identity with (5.4) we get

$$E(u(c_n); 0, \rho_n) = o_n(1) \text{ as } n \to \infty.$$

As a consequence, localizing $u(c_n)$ by a smooth cutoff function $\chi \in C_c^\infty(B_2(0))$ we have that $v_n = u(c_n)\chi_{\rho_n/2}$ satisfies $E(v_n) = o_n(1)$. However, we will argue that $\|v_n\|_{\mathcal{E}} \simeq \eta$ which by

Lemma 2.12 will yield a contradiction. To this end, we find parameters $\vec{\sigma}_n, \vec{\mu}_n$ such that

$$\eta \simeq \mathbf{d}\left(c_n\right) \simeq \left(\left\|u\left(c_n\right)-u^*\left(c_n\right)-\mathcal{W}\left(\vec{\sigma}_n, \vec{\mu}_n\right)\right\|_{\mathcal{E}}^2+\sum_{j=0}^N\left(\frac{\mu_{n, j}}{\mu_{n, j+1}}\right)^{\frac{D-2}{2}}\right)^{\frac{1}{2}}. \tag{5.6}$$

Then using the above decomposition and (5.5) as inputs to Lemma 2.7 with η possibly small enough, we get that $\vec{\sigma}_n=\vec{\iota}_n$, $\rho_n \ll \mu_{n,1} \ll \ldots \mu_{n,N} \ll \rho\left(c_n\right) \ll c_n$ and $\left|\lambda_{n,j}/\mu_{n,j}-1\right| \lesssim \theta(\eta)$ for $\theta(\eta)$ as in Lemma 2.7. Therefore (5.5) implies

$$\left\|u\left(c_n\right)-u^*\left(c_n\right)-\mathcal{W}\left(\vec{\sigma}_n, \vec{\mu}_n\right)\right\|_{\mathcal{E}\left(\rho_n, \infty\right)}^2+\sum_{j=0}^N\left(\frac{\mu_{n, j}}{\mu_{n, j+1}}\right)^{\frac{D-2}{2}}=o_n(1). \tag{5.7}$$

As the bubbles are confined between the scales ρ_n and $\rho(c_n)$, i.e. $\rho_n \ll \mu_{n,1}$ and $\mu_{n,N} \ll \rho\left(c_n\right)$ we get

$$\left\|u^*\left(c_n\right)+\mathcal{W}\left(\vec{\iota}_n, \vec{\mu}_n\right)\right\|_{\mathcal{E}\left(0, \rho_n\right)}=o_n(1). \tag{5.8}$$

From (5.6), (5.7) and (5.8) we deduce that $\|v_n\|_{\mathcal{E}}^2 \simeq \eta$ which yields a contradiction and thus $K \geq 1$. □

Remark 5.7. Without loss of generality we assume that $\mathbf{d}(a_n)=\epsilon_n$, $\mathbf{d}(b_n)=\eta$, and $\mathbf{d}(t) \in [\epsilon_n, \eta]$ for each $t \in [a_n, b_n]$. Furthermore, given parameters ϵ_n and η we will often enlarge ϵ_n by choosing another sequence $\epsilon_n \leq \widetilde{\epsilon}_n \to 0$, and $0<\widetilde{\eta} \leq \eta$, resulting in a smaller collision $[\widetilde{a}_n, \widetilde{b}_n] \subset [a_n, b_n] \cap \mathcal{C}_K(\widetilde{\eta}, \widetilde{\epsilon}_n)$.

5.2. **Decomposition of the solution.**

Lemma 5.8. *Let $K \geq 1$ be the number given by Lemma 5.6, and let η, ϵ_n, a_n and b_n be some choice of objects satisfying the requirements of Definition 5.4. Then there exists a sequence $\vec{\sigma}_n \in\{-1,1\}^{N-K}$, a function $\vec{\mu}=\left(\mu_{K+1}, \ldots, \mu_N\right) \in C^1\left(\cup_{n \in \mathbb{N}}\left[a_n, b_n\right] ;(0, \infty)^{N-K}\right)$, a sequence $\nu_n \to 0$, and a sequence $m_n \in \mathbb{Z}$, so that defining the function,*

$$\nu: \cup_{n \in \mathbb{N}}\left[a_n, b_n\right] \rightarrow(0, \infty), \quad \nu(t):=\nu_n \mu_{K+1}(t), \tag{5.9}$$

we have,

$$\lim_{n \rightarrow \infty} \sup_{t \in\left[a_n, b_n\right]}\left(\mathbf{d}_K(t ; \nu(t))+\|u(t)\|_{\mathcal{E}(\nu(t) \leq r \leq 2 \nu(t))}\right)=0,$$

and defining $g(t)$ for $t \in \cup_n\left[a_n, b_n\right]$ by

$$\left(1-\chi_{\nu(t)}\right)\left(u(t)-u^*\right)=\sum_{j=K+1}^N \sigma_{n, j} W_{\mu_j(t)}+g(t),$$

we have, $g(t) \in \mathcal{E}$, and

$$\lim_{n \rightarrow \infty} \sup_{t \in\left[a_n, b_n\right]}\left(\|g(t)\|_{\mathcal{E}}^2+\left(\frac{\nu(t)}{\mu_{K+1}(t)}\right)^{\frac{D-2}{2}}+\sum_{j=K+1}^N\left(\frac{\mu_j(t)}{\mu_{j+1}(t)}\right)^{\frac{D-2}{2}}\right)=0, \tag{5.10}$$

with the convention that $\mu_{N+1}(t)=t$. Finally, $\nu(t)$ satisfies the estimate,

$$\lim_{n \rightarrow \infty} \sup_{t \in\left[a_n, b_n\right]}\left|\nu^{\prime}(t)\right|=0.$$

Remark 5.9. The scale $\nu(t)$ separates the $N-K$ "exterior" bubbles defined on the collision intervals $[a_n, b_n]$ from the K "interior" bubbles. This is evidenced from (5.10).

Proof. The proof proceeds by observing that the definition of $\mathbf{d}_K$ yields the existence of signs $\vec{\sigma}$ and scales $\vec{\mu}$. Finally the scale $\nu(t) = \nu_n \mu_{K+1}(t)$ is chosen by finding a sequence $\nu_n \to 0$ such that

$$\begin{aligned} &\rho_K(t) \leq \nu_n \mu_{K+1}(t) \ll \mu_{K+1}(t) \\ &\lim_{n\to\infty} \sup_{t\in[a_n,b_n]} \|u(t) - u^*\|_{\mathcal{E}\left(\frac{1}{4}\nu_n\mu_{K+1}(t)\leq 4\nu_n\mu_{K+1}(t)\right)} = 0. \end{aligned}$$

See proof of Lemma 5.9 [JL23b] for details. □

Lemma 5.10 (Basic modulation)**.** *There exist $C_0, \eta_0 > 0$ such that the following is true. Let $J \subset [a_n, b_n]$ be an open time interval such that $\mathbf{d}(t) \leq \eta_0$ for all $t \in J$. Then, there exist $\vec{\iota} \in \{-1, 1\}^K$ (independent of $t \in J$), modulation parameters $\vec{\lambda} \in C^1(J; (0, \infty)^K)$, and $g(t) \in \mathcal{E}$ satisfying, for all $t \in J$,*

$$u^*(t) := \begin{cases} \left(1 - \chi_{\nu(t)}\right) u(t) & \text{if } T_+ < \infty \\ 0 & \text{if } T_+ = \infty \end{cases}$$

$$g(t) = u(t) - u^*(t) - \mathcal{W}(\vec{\iota}, \vec{\lambda}(t)) \tag{5.11}$$

$$\left\langle \mathcal{Z}_{\underline{\lambda_j(t)}} \mid g(t) \right\rangle = 0, \tag{5.12}$$

and $\nu(t)$ is as in (5.9). The estimates,

$$C_0^{-1}\mathbf{d}(t) \leq \|g(t)\|_{\mathcal{E}} + \sum_{j=1}^{K-1} \left(\frac{\lambda_j(t)}{\lambda_{j+1}(t)} \right)^{\frac{D-2}{4}} \leq C_0 \mathbf{d}(t), \tag{5.13}$$

$$\left|\lambda_j'(t)\right| \leq \frac{C_0}{\lambda_j(t)} \mathbf{d}(t), \tag{5.14}$$

$$\|g(t)\|_{\mathcal{E}} + \sum_{j\notin\mathcal{S}} \left(\frac{\lambda_j(t)}{\lambda_{j+1}(t)} \right)^{\frac{D-2}{4}} \leq C_0 \max_{j\in\mathcal{S}} \left(\frac{\lambda_j(t)}{\lambda_{j+1}(t)} \right)^{\frac{D-2}{4}} + \max_{1\leq i\leq K} \left|a_i^-(t)\right|, \tag{5.15}$$

$$\left|\frac{\mathrm{d}}{\mathrm{d}t} a_j^-(t)\right| \leq \frac{C_0}{\lambda_j(t)^2} \mathbf{d}(t)^2, \text{ when } D \geq 6 \tag{5.16}$$

where $\mathcal{S} := \{j \in \{1, \ldots, K-1\} : \iota_j = \iota_{j+1}\}$ and by convention $\lambda_0(t) = 0, \lambda_{K+1}(t) = \infty$ for all $t \in J$ and $j \in \{1, \cdots, K\}$.

Proof. The proof of (5.11), (5.12) and (5.13) and (5.15) follows directly from the argument outlined in Step 1 of proof of Lemma 5.12 in [JL23b]. Thus it suffices to prove the dynamical estimate (5.14) and (5.16). Differentiating in time the orthogonality conditions (5.12) yields, for each $j = 1, \ldots, N$, the identity,

$$0 = -\frac{\lambda_j'}{\lambda_j} \left\langle \underline{\Lambda} \mathcal{Z}_{\underline{\lambda_j}} \mid g \right\rangle + \left\langle \mathcal{Z}_{\underline{\lambda_j}} \mid \partial_t g \right\rangle. \tag{5.17}$$

Next, differentiating in time the expression for $g(t)$ in (5.11)

$$\begin{aligned}
\partial_t g &= \chi_\nu \partial_t u(t) - u(t)\frac{\nu'(t)}{\nu(t)}\left(r\partial_r \chi\right)\left(\cdot/\nu(t)\right) + \sum_{j=1}^N \iota_j \lambda_j' \Lambda W_{\underline{\lambda_j}} \\
&= \Delta g + f'(\mathcal{W}(\vec{\iota}, \vec{\lambda}))g + f(\mathcal{W}(\vec{\iota}, \vec{\lambda})) - \sum_{j=1}^N \iota_j f(W_{\underline{\lambda_j}}) \\
&\quad + f(\mathcal{W}(\vec{\iota}, \vec{\lambda}) + g) - f(\mathcal{W}(\vec{\iota}, \vec{\lambda})) - f'(\mathcal{W}(\vec{\iota}, \vec{\lambda}))g + \sum_{j=1}^N \iota_j \lambda_j' \Lambda W_{\underline{\lambda_j}} \\
&\quad \left(-u\Delta\chi_\nu - 2\partial_r u \partial_r \chi_\nu - u(t)\frac{\nu'(t)}{\nu(t)}\left(r\partial_r\chi\right)\left(\cdot/\nu(t)\right) + f(u)\chi_\nu - f(\chi_\nu u)\right) \\
&= -\mathcal{L}_{\mathcal{W}} g + f_{\mathbf{i}}(\vec{\iota}, \vec{\lambda}) + f_{\mathbf{q}}(\vec{\iota}, \vec{\lambda}, g) + \phi(u, \nu) + \sum_{j=1}^N \iota_j \lambda_j' \Lambda W_{\underline{\lambda_j}}.
\end{aligned} \tag{5.18}$$

where

$$\mathcal{L}_{\mathcal{W}} := -\Delta - f'(\mathcal{W}(\vec{\iota}, \vec{\lambda})), \quad f_{\mathbf{i}}(\vec{\iota}, \vec{\lambda}) := f\big(\mathcal{W}(\vec{\iota}, \vec{\lambda})\big) - \sum_{j=1}^N \iota_j f(W_{\lambda_j})$$

$$f_{\mathbf{q}}(\vec{\iota}, \vec{\lambda}, g) := f\big(\mathcal{W}(\vec{\iota}, \vec{\lambda}) + g\big) - f\big(\mathcal{W}(\vec{\iota}, \vec{\lambda})\big) - f'\big(\mathcal{W}(\vec{\iota}, \vec{\lambda})\big)g$$

$$\phi(u, \nu) := -u\Delta\chi_\nu - 2\partial_r u \partial_r \chi_\nu - u(t)\frac{\nu'(t)}{\nu(t)}\left(r\partial_r\chi\right)\left(\cdot/\nu(t)\right) + f(u)\chi_\nu - f(\chi_\nu u)$$

The subscript **i** above stands for "interaction" and **q** stands for "quadratic." For each $j \in \{1, \dots, N\}$ we pair (5.18) with $\mathcal{Z}_{\underline{\lambda_j}}$ and use (5.17) to obtain the following system

$$\begin{aligned}
&\iota_j \lambda_j' \Big(\big\langle \Lambda W \mid \mathcal{Z} \big\rangle - \frac{\iota_j}{\lambda_j}\big\langle \mathcal{Z}_{\underline{\lambda_j}} \mid g \big\rangle\Big) + \sum_{i \neq j} \iota_i \lambda_i' \big\langle \Lambda W_{\underline{\lambda_i}} \mid \mathcal{Z}_{\underline{\lambda_j}} \big\rangle \\
&\qquad = \big\langle \mathcal{L}_{\mathcal{W}} g \mid \mathcal{Z}_{\underline{\lambda_j}} \big\rangle - \big\langle f_{\mathbf{i}}(\vec{\iota}, \vec{\lambda}) \mid \mathcal{Z}_{\underline{\lambda_j}} \big\rangle - \big\langle f_{\mathbf{q}}(\vec{\iota}, \vec{\lambda}, g) \mid \mathcal{Z}_{\underline{\lambda_j}} \big\rangle - \big\langle \phi(u, \nu) \mid \mathcal{Z}_{\underline{\lambda_j}} \big\rangle.
\end{aligned}$$

The above is diagonally dominant for all sufficiently small $\eta > 0$, hence invertible. Then estimating each term

$$\begin{aligned}
&\Big|\big\langle \mathcal{L}_{\mathcal{W}} g \mid \mathcal{Z}_{\underline{\lambda_j}} \big\rangle\Big| \lesssim \frac{1}{\lambda_j}\|g\|_{\mathcal{E}} \\
&\Big|\big\langle f_{\mathbf{i}}(\vec{\iota}, \vec{\lambda}) \mid \mathcal{Z}_{\underline{\lambda}_j} \big\rangle\Big| \lesssim \frac{1}{\lambda_j}\Big(\Big(\frac{\lambda_j}{\lambda_{j+1}}\Big)^{\frac{D-2}{2}} + \Big(\frac{\lambda_{j-1}}{\lambda_j}\Big)^{\frac{D-2}{2}}\Big) \lesssim \frac{1}{\lambda_j}(\mathbf{d}(t)^2 + o_n(1)). \\
&\Big|\big\langle f_{\mathbf{q}}(\vec{\iota}, \vec{\lambda}, g) \mid \mathcal{Z}_{\underline{\lambda}_j} \mid \big\rangle\Big| \lesssim \frac{1}{\lambda_j}(\mathbf{d}(t)^2 + o_n(1)), \quad \Big|\big\langle \phi(u, \nu) \mid \mathcal{Z}_{\underline{\lambda_j}} \big\rangle\Big| \lesssim \frac{1}{\lambda_j} o_n(1).
\end{aligned}$$

It follows that,

$$\left|\lambda_j'\right| \lesssim \frac{1}{\lambda_j}\Big(\mathbf{d}(t) + \zeta_{3,n}\Big)$$

for some sequence $\zeta_{3,n} \to 0$ as $n \to \infty$. Then (5.14) follows by enlarging ϵ_n.
Step 2:(Proof of (5.16)). Denote

$$\alpha_{\lambda_j}^- = \frac{\kappa}{\lambda_j}\mathcal{Y}_{\underline{\lambda_j}}.$$

Then using this notation and differentiating (2.3) we get

$$\frac{\mathrm{d}}{\mathrm{d}t}a_j^- = \left\langle \partial_t \alpha_{\lambda_j}^- \mid g \right\rangle + \left\langle \alpha_{\lambda_j}^- \mid \partial_t g \right\rangle.$$

Expanding the first term on the right gives,

$$\begin{aligned}\left\langle \partial_t \alpha_{\lambda_j}^- \mid g \right\rangle &= \kappa \left\langle \partial_t \left(\lambda_j^{-1} \mathcal{Y}_{\underline{\lambda_j}} \right) \mid g \right\rangle \\ &= -\kappa \frac{\lambda_j'}{\lambda_j} \left\langle \lambda_j^{-1} \mathcal{Y}_{\underline{\lambda_j}} + \frac{1}{\lambda_j} (\underline{\Lambda} \mathcal{Y})_{\underline{\lambda_j}} \middle| \, g \right\rangle\end{aligned}$$

and thus using (5.14) we get

$$\left| \left\langle \partial_t \alpha_{\lambda_j}^- \mid g \right\rangle \right| \lesssim \frac{1}{\lambda_j^2} \left(\mathbf{d}(t)^2 + o_n(1) \right).$$

For the second term using (5.18) and the notation used in the previous step

$$\begin{aligned}\left\langle \alpha_{\lambda_j}^- \mid \partial_t g \right\rangle = &-\left\langle \alpha_{\lambda_j}^- \mid \mathcal{L}_{\mathcal{W}} g \right\rangle + \left\langle \alpha_{\lambda_j}^- \mid f_{\mathbf{i}}(\vec{\iota}, \vec{\lambda}) \right\rangle + \left\langle \alpha_{\lambda_j}^- \mid f_{\mathbf{q}}(\vec{\iota}, \vec{\lambda}, g) \right\rangle \\ &+ \left\langle \alpha_{\lambda_j}^- \mid \phi(u, \nu) \right\rangle + \left\langle \alpha_{\lambda_j}^- \mid \sum_{i=1}^{N} \iota_i \lambda_i' \Lambda W_{\underline{\lambda_i}} \right\rangle.\end{aligned}$$

Then estimating as before

$$\begin{aligned}&\left| \left\langle \alpha_{\lambda_j}^- \mid -\mathcal{L}_{\mathcal{W}} g \right\rangle \right| \lesssim \frac{1}{\lambda_j^2} (\mathbf{d}(t) + o_n(1)) \\ &\left| \left\langle \alpha_{\lambda_j}^- \mid f_{\mathbf{i}}(\vec{\iota}, \vec{\lambda}) \right\rangle \right| \lesssim \frac{1}{\lambda_j^2} \Big(\Big(\frac{\lambda_j}{\lambda_{j+1}} \Big)^{\frac{D-2}{2}} + \Big(\frac{\lambda_{j-1}}{\lambda_j} \Big)^{\frac{D-2}{2}} \Big) \lesssim \frac{1}{\lambda_j^2} (\mathbf{d}(t)^2 + o_n(1)). \\ &\left| \left\langle \alpha_{\lambda_j}^- \mid f_{\mathbf{q}}(\vec{\iota}, \vec{\lambda}, g) \right\rangle \right| \lesssim \frac{1}{\lambda_j^2} (\mathbf{d}(t)^2 + o_n(1)), \quad \left| \left\langle \alpha_{\lambda_j}^- \mid \phi(u, \nu) \mid \right\rangle \right| \lesssim \frac{1}{\lambda_j^2} o_n(1).\end{aligned}$$

For the last term note that using $\langle \mathcal{Y} \mid \Lambda W \rangle = 0$ we get

$$\left\langle \alpha_{\lambda_j}^- \mid \partial_t \mathcal{W}(\vec{\iota}, \vec{\lambda}) \right\rangle = \sum_{i \neq j} \iota_i \kappa \frac{\lambda_i'}{\lambda_j} \left\langle \mathcal{Y}_{\underline{\lambda_j}} \mid \Lambda W_{\underline{\lambda_i}} \right\rangle.$$

Using the estimates,

$$\left| \left\langle \mathcal{Y}_{\underline{\lambda_j}} \mid \Lambda W_{\underline{\lambda_i}} \right\rangle \right| \lesssim \begin{cases} \left(\frac{\lambda_i}{\lambda_j} \right)^{\frac{D-4}{2}} & \text{if } i < j \\ \left(\frac{\lambda_i}{\lambda_j} \right)^{\frac{D}{2}} & \text{if } i > j \end{cases}$$

and the fact that $D \geq 6$, we obtain,

$$\left| \left\langle \alpha_{\lambda_j}^- \mid \partial_t \mathcal{W}(\vec{\iota}, \vec{\lambda}) \right\rangle \right| \lesssim \frac{1}{\lambda_j^2} \left(\mathbf{d}(t)^2 + o_n(1) \right).$$

Combining all the estimates above implies (5.16). □

Next, we prove a lemma connecting localized bubbling as in Lemma 1.6 to sequential bubbling.

Lemma 5.11. *There exists a constant $\eta_0 > 0$ having the following property. Let $t_n \in [a_n, b_n]$ and let μ_n be a positive sequence satisfying the conditions:*

(1) $\lim_{n \to \infty} \frac{\mu_n}{\mu_{K+1}(t_n)} = 0$,
(2) $\mu_n \geq \nu(t_n)$ *or* $\|u(t_n)\|_{\mathcal{E}(\mu_n, \nu(t_n))} \leq \eta_0$
(3) $\lim_{n \to \infty} \delta_{\mu_n}(t_n) = 0$.

Then $\lim_{n\to\infty} \mathbf{d}(t_n) = 0$.

Proof. Let R_n be a sequence such that $\mu_n \ll R_n \ll \mu_{K+1}(t_n)$. Without loss of generality, we can assume $R_n \geq \nu(t_n)$, since it suffices to replace R_n by $\nu(t_n)$ for all n such that $R_n < \nu(t_n)$. Let $M_n, \vec{\iota}_n, \vec{\lambda}_n$ be parameters such that

$$\left\| u(t_n) - \mathcal{W}\left(\vec{\iota}_n, \vec{\lambda}_n\right) \right\|^2_{\mathcal{E}(r\leq\mu_n)} + \sum_{j=1}^{M_n-1} \left(\frac{\lambda_{n,j}}{\lambda_{n,j+1}}\right)^{\frac{D-2}{2}} \to 0 \tag{5.19}$$

which exist by the definition of the localized distance function (1.3). Set

$$u_n^{(i)} := \chi_{\frac{1}{2}\mu_n} u(t_n), \quad u_n^{(o)} := (1-\chi_{R_n}) u(t_n), \quad u_n^{(m)} := u(t_n) - u_n^{(i)} - u_n^{(o)}.$$

Observe that if μ_n is a positive sequence such that $\lim_{n\to\infty} \delta_{\mu_n}(t_n) = 0$, then

$$\lim_{n\to\infty} \| u(t_n) \|_{\mathcal{E}\left(\frac{1}{2}\mu_n, \mu_n\right)} = 0. \tag{5.20}$$

Combining this with the localized decomposition (5.19) we have

$$\lim_{n\to\infty} \|u_n^{(i)} - \mathcal{W}(\vec{\iota}_n, \vec{\lambda}_n)\|_{\mathcal{E}} = 0.$$

Furthermore observe that if $t_n \in [a_n, b_n]$ and $\nu(t_n) \leq R_n \ll \mu_{K+1}(t_n)$, then

$$\lim_{n\to\infty} \|u(t_n)\|_{\mathcal{E}(R_n, 2R_n)} = 0. \tag{5.21}$$

Thus using the second assumption along with (5.20) and (5.21) for n large enough we have

$$\|u_n^{(m)}\|_{\mathcal{E}} \leq 2\eta_0,$$

which implies from Lemma 2.12 that $0 \leq E(u_n^{(m)}) \leq 2\eta_0$. We also have, again using (5.20) and (5.21),

$$\limsup_{n\to\infty} |E(u(t_n)) - E(u_n^{(i)}) - E(u_n^{(m)}) - E(u_n^{(o)})| = 0.$$

Combining the above convergence with $\lim_{n\to\infty} E(u_n^{(o)}) = (N-K)E(W) + E(u^*)$ we see that $M_n = K$ and $\lim_{n\to\infty} E(u_n^{(m)}) = 0$. Using Sobolev embedding, we get $\lim_{n\to\infty} \|u_n^{(m)}\|_{\mathcal{E}} = 0$ which establishes the desired result. □

6. Conclusion of the Proof

Now we are in a position to conclude the proof of Theorem 1.1. The idea is to first observe that the length of the collision interval is related to the scale of the K-th bubble. This follows from the dynamical estimate (5.14) and the fact that K is minimal as in Definition 5.5. As a consequence, one can identify a suitable sub-interval of the collision interval in which the scale λ_K does not change much. Finally combining the previous observation along with the fact that the tension is finite one can deduce a contradiction, which in particular implies that (5.2) and thus Theorem 1.1 holds.

For convenience, we assume that whenever $[a_n, b_n] \in \mathcal{C}_K(\epsilon_n, \eta)$ then $\mathbf{d}(a_n) = \epsilon_n$, $\mathbf{d}(b_n) = \eta$ and $\mathbf{d}(t) \in [\epsilon_n, \eta]$ for all $t \in [a_n, b_n]$. This can always be done by Remark 5.7.

Lemma 6.1. *If $\eta_0 > 0$ is small enough, then for any $\eta \in (0, \eta_0]$ there exist $\epsilon \in (0, \eta)$ and $C_u > 0$ with the following property. If $[c, d] \subset [a_n, b_n]$, $\mathbf{d}(c) \leq \epsilon$ and $\mathbf{d}(d) \geq \eta$, then,*

$$(d-c)^{\frac{1}{2}} \geq C_u^{-1} \lambda_K(c)$$

Proof. If not, then there exists $\eta > 0$, sequences $\epsilon_n \to 0$, $[c_n, d_n] \subset [a_n, b_n]$, and $C_n \to \infty$ so that $\mathbf{d}(c_n) \leq \epsilon_n$, $\mathbf{d}(d_n) \geq \eta$ and

$$(d_n - c_n)^{\frac{1}{2}} \leq C_n^{-1} \lambda_K(c_n). \tag{6.1}$$

We will show that $[c_n, d_n] \in \mathcal{C}_{K-1}(\epsilon_n, \eta)$, hence contradicting the minimality of K.
First, using (5.14)

$$\left|\lambda_j(t)^2 - \lambda_j(c_n)^2\right| \leq C_0(t - c_n) \tag{6.2}$$

for all $t \in [c_n, d_n]$ and all $j = 1, \cdots, N$. Hence, using the contradiction assumption (6.1) we can ensure that for large enough n,

$$\frac{3}{4} \leq \frac{\lambda_j(t)}{\lambda_j(c_n)} \leq \frac{5}{4}$$

for all $j = K, \ldots, N$ and all $t \in [c_n, d_n]$. Since $\mathbf{d}(c_n) \to 0$ we have

$$\lim_{n\to\infty} \sup_{t\in[c_n,d_n]} \sum_{j=K}^{N} \Big(\frac{\lambda_j(t)}{\lambda_{j+1}(t)}\Big)^{\frac{D-2}{2}} = 0 \tag{6.3}$$

and furthermore there exists a sequence (r_n) such that

$$\lambda_{K-1}(c_n) + (d_n - c_n)^{\frac{1}{2}} \ll r_n \ll \lambda_K(c_n) \quad \text{and} \quad \lim_{n\to\infty} \widetilde{E}(u(c_n); \frac{1}{8}r_n, 8r_n) = 0. \tag{6.4}$$

Letting $\phi(r)$ be a smooth bump equal to 1 for $r \in (1/4, 4)$ and supported for $r \in (1/8, 8)$ with $|\phi'(r)| \leq \tilde{C}$, we apply (2.7) with $\phi(\cdot/r_n)$ we deduce that for any $t \in [c_n, d_n]$,

$$\begin{aligned}
&\int_{\mathbb{R}_+} \tilde{\mathbf{e}}(u(t))\phi^2 - \int_{\mathbb{R}_+} \tilde{\mathbf{e}}(u(c_n))\phi^2 \\
&\leq 4\int_{c_n}^{t}\int_{\mathbb{R}_+} |\partial_r u|^2 |\partial_r \phi|^2 + 2\left(\int_{c_n}^{t}\int_{\mathbb{R}_+} |u|^{2p}\phi^2\right)^{1/2} \left(\int_{c_n}^{t}\int_{\mathbb{R}_+} (\partial_t u)^2\phi^2\right)^{1/2} \\
&\quad + 2\left(\int_{c_n}^{t}\int_{\mathbb{R}_+} \frac{|u|^2}{r^4}\phi^2\right)^{1/2} \left(\int_{c_n}^{t}\int_{\mathbb{R}_+} (\partial_t u)^2\phi^2\right)^{1/2}
\end{aligned}$$

which implies that

$$\widetilde{E}(u(t); \frac{1}{4}r_n, 4r_n) \leq \widetilde{E}(u(c_n); 1/8r_n, 8r_n) + C_0\frac{d_n - c_n}{r_n^2} + C_1\frac{(d_n - c_n)^{1/2}}{r_n}$$

and hence,

$$\lim_{n\to\infty} \sup_{t\in[c_n,d_n]} \widetilde{E}(u(t); \frac{1}{4}r_n, 4r_n) = 0.$$

Next, we claim that

$$\sup_{t\in[c_n,d_n]} \widetilde{E}(u(t); \frac{1}{4}r_n, \infty) \leq (N - (K-1))\widetilde{E}(W) + o_n(1) \tag{6.5}$$

In the case $T_+ < \infty$ we recall that $\alpha(t) = \alpha_n (T_+ - t)^{\frac{1}{2}}$ and we write,

$$\widetilde{E}\left(u(t) - u^*(t); \frac{1}{4}r_n, \infty\right) = \widetilde{E}\left(u(t) - u^*(t); \frac{1}{4}r_n, \frac{1}{4}\alpha(t)\right) + \widetilde{E}\left(u(t) - u^*(t); \frac{1}{4}\alpha(t), \infty\right)$$

Since $\alpha(t) \geq \rho(t)$ we have,

$$\lim_{t\to\infty} \widetilde{E}\left(u(t) - u^*(t); \frac{1}{4}\alpha(t), \infty\right) = 0$$

Recalling that $u(t,r) - u^*(t,r) = u(t,r)$ for all $r \leq \alpha(t)$ we again apply (2.7) with the cut-off function $\phi(t,r) = (1 - \chi_{4r_n}(r))\, \chi_{\frac{1}{4}\alpha(t)}(r)$. Since $\frac{\mathrm{d}}{\mathrm{d}t}\phi(t,r) \leq 0$ we use (2.7) to deduce that for all $t \in [c_n, d_n]$,

$$\widetilde{E}\left(u(t) - u^*(t); \frac{1}{4}r_n, \frac{1}{4}\alpha(t)\right) \leq \widetilde{E}\left(u\left(c_n\right) - u^*\left(c_n\right); \frac{1}{8}r_n, \frac{1}{2}\alpha(t)\right) + C_0\frac{d_n - c_n}{r_n^2} + C_1\frac{(d_n - c_n)^{1/2}}{r_n}$$

and the right hand side tends to zero as $n \to \infty$, proving (6.5) in the case $T_+ < \infty$. When $T_+ = \infty$, recall that $u^*(t) := 0$ and $\alpha(t) = \alpha_n\sqrt{t}$. Thus we have

$$\widetilde{E}(u(t); \frac{1}{4}r_n, \infty) = \widetilde{E}(u(t); \frac{1}{4}r_n, \frac{1}{4}\alpha(t)) + \widetilde{E}(u(t); \frac{1}{4}\alpha(t), \infty)$$

Using (4.2) we see that both quantities on the RHS tend to zero as $n \to \infty$, proving (6.5) in the case when $T_+ = \infty$.
Now using (6.2) with $j = K - 1$ we get

$$\sup_{t\in[c_n,d_n]} \left|\lambda_{K-1}(t)^2 - \lambda_{K-1}(c_n)^2\right| \lesssim d_n - c_n,$$

which in turn implies

$$\sup_{t\in[c_n,d_n]} \frac{\lambda_{K-1}(t)}{r_n} \lesssim \frac{\lambda_{K-1}(c_n)}{r_n} + \frac{(d_n - c_n)^{\frac{1}{2}}}{r_n} \to 0 \quad \text{as} \quad n \to \infty$$

since r_n satisfies (6.4). As a consequence if we denote

$$v(t) := (1 - \chi_{r_n})(u(t) - u^*)$$

with $v(t) \in \mathcal{E}$ for $t \in [c_n, d_n]$, then

$$\|v(t) - \mathcal{W}(\iota_K, \ldots, \iota_N, \lambda_K(t), \ldots \lambda_N(t))\|_{\mathcal{E}} + \sum_{j=K}^{N-1} \Big(\frac{\lambda_j(t)}{\lambda_{j+1}(t)}\Big)^{\frac{D-2}{4}} \lesssim \eta.$$

Thus $\mathbf{d}_{N-K+1}(v(t)) \lesssim \eta$. Applying Lemma 2.6 we can find signs $\vec{\iota} \in \{-1,1\}^{N-K+1}$, scales $\widetilde{\lambda}_K(t), \ldots, \widetilde{\lambda}_N(t)$ and error term $h(t)$ such that

$$h(t) = v(t) - \mathcal{W}(\iota_K, \ldots, \iota_N, \widetilde{\lambda}_K(t), \ldots, \widetilde{\lambda}_N(t))$$

so that

$$0 = \left\langle \mathcal{Z}_{\underline{\widetilde{\lambda}_j(t)}} \mid h(t)\right\rangle, \quad \text{and} \quad \|h(t)\|_{\mathcal{E}} + \sum_{j=K}^{N-1} \Big(\frac{\widetilde{\lambda}_j(t)}{\widetilde{\lambda}_{j+1}(t)}\Big)^{\frac{D-2}{4}} \lesssim \eta.$$

Next, using (6.3) and the fact that $\widetilde{\lambda}_j(t)$ satisfy $|\widetilde{\lambda}_j(t)/\lambda_j(t) - 1| \lesssim \eta$, we get

$$\lim_{n\to\infty} \sup_{t\in[c_n,d_n]} \sum_{j=K}^{N-1} \Big(\frac{\widetilde{\lambda}_j(t)}{\widetilde{\lambda}_{j+1}(t)}\Big)^{\frac{D-2}{4}} = 0. \tag{6.6}$$

Next, using Lemma 2.17 with r_n, a sequence R_n satisfying $\widetilde{\lambda}_K(c_n) \ll R_n \ll \widetilde{\lambda}_{K+1}(c_n)$ and $u_n(t) = v(c_n + t)$ we get

$$\lim_{n\to\infty} \sup_{t\in[c_n,d_n]} \|h(t)\|_{\mathcal{E}(r_n,R_n)} = 0.$$

To show that $\lim_{n\to\infty}\sup_{t\in[c_n,d_n]}\|h(t)\|_{\mathcal{E}(r_n,\infty)} = 0$ one can first consider a partition of the interval $[r_n/2,\infty)$ in terms of overlapping intervals

$$[r_n/2,\infty) = \bigcup_{j=K}^{N}[r_n^j/2, 2R_n^j) = \bigcup_{j=K}^{N} I_n^j,$$

where the sequences satisfy $\frac{r_n^j}{2} \ll \widetilde{\lambda}_j(c_n) \ll \frac{r_n^{j+1}}{2} < r_n^{j+1} = R_n^j < 2R_n^j$ and $\frac{r_n^N}{2} \ll \widetilde{\lambda}_N(c_n) \ll R_n^N$ for $j = K,\cdots,N-1$ with $r_n^K = r_n$ and $R_n^N = +\infty$. Then observe that

$$[r_n,\infty) = \bigcup_{j=K}^{N}[r_n^j, R_n^j) = \bigcup_{j=K}^{N}\widetilde{I}_n^j.$$

For each $j = K,\cdots,N$, we can apply Lemma 2.17 to each interval I_n^j with the observation that on this interval h is small since u is close to a single bubble with scale $\widetilde{\lambda}_j$ as $\mathbf{d}(c_n) \le \epsilon_n$. Propagating this smallness bound we deduce that v remains to close to this bubble on $\widetilde{I}_n^j$. Summing this up we see that h is close to the multi-scale bubble configuration on $\bigcup_{j=K}^{N}\widetilde{I}_n^j = [r_n,\infty)$ and thus

$$\lim_{n\to\infty}\sup_{t\in[c_n,d_n]}\|h(t)\|_{\mathcal{E}(r_n,\infty)} = 0.$$

Therefore, setting $\rho_{K-1}(t) := r_n$ for $t\in[c_n,d_n]$ we get

$$\lim_{n\to\infty}\sup_{t\in[c_n,d_n]}\mathbf{d}(t;\rho_{K-1}(t)) = 0$$

which means that we can find $\widetilde{\eta} > 0$, $\widetilde{\epsilon}_n \to 0$ such that $[c_n,d_n]\in\mathcal{C}_{K-1}(\widetilde{\epsilon}_n,\widetilde{\eta})$ contradicting the minimality of K. □

Remark 6.2. When $D \ge 6$, one could alternatively proceed in the following manner after (6.6). From (2.4) we get

$$\|h(t)\|_{\mathcal{E}}^2 \lesssim \sum_{j=K}^{N-1}\left(\frac{\tilde{\lambda}_j(t)}{\tilde{\lambda}_{j+1}(t)}\right)^{\frac{D-2}{2}} + \max_{i\in\{K,\ldots,N\}}|a_i^-|^2(t) + o_n(1).$$

Using (5.16) we deduce that

$$|a_j^-(t)| \le |a_j^-(c_n)| + \int_{c_n}^{d_n}\left|\frac{\mathrm{d}a_j^-}{\mathrm{d}t}\right|dt \lesssim |a_j^-(c_n)| + \frac{d_n - c_n}{\lambda_K(c_n)^2} \to 0$$

as $n\to\infty$ since $(d_n - c_n)^{1/2} \ll \lambda_K(c_n)$ and $|a_j^-(c_n)| \lesssim \mathbf{d}(c_n) \le \epsilon_n$. By using the analysis carried out in Appendix A and B in [JL23b] this type of argument can also be carried out in dimension $D \ge 4$.

Lemma 6.3. *Let $\eta_0 > 0$ be as in Lemma 6.1, $\eta\in(0,\eta_0]$, $\epsilon_n\to 0$ be some sequence, and let $[a_n,b_n]\in\mathcal{C}_K(\epsilon_n,\eta)$. Then, there exist $\epsilon\in(0,\eta)$, $n_0\in\mathbb{N}$, and $c_n,d_n\in(a_n,b_n)$ such that for all $n\ge n_0$, we have*

$$\mathbf{d}(t) \ge \epsilon, \quad \forall\ t\in[c_n,d_n], \tag{6.7}$$

$$d_n - c_n = \frac{1}{n}\lambda_K(c_n)^2, \tag{6.8}$$

and

$$\frac{1}{2}\lambda_K(c_n) \le \lambda_K(t) \le 2\lambda_K(c_n) \quad \forall\ t\in[c_n,d_n]. \tag{6.9}$$

Proof. See proof of Lemma 6.2 in [JL23a]. □

Proof of Theorem 1.1. We proceed by a contradiction argument. Suppose that Theorem 1.1 is false. Then let $[a_n, b_n] \in \mathcal{C}_K(\epsilon_n, \eta)$ be a sequence of disjoint collision intervals as in Lemma 5.10, and let $\eta > 0$ be sufficiently small such that Lemma 6.1 and Lemma 5.11 hold. Let $\epsilon > 0$, n_0, and $[c_n, d_n]$ be as in Lemma 6.3.
Then first we will show that there exists a constant $c_0 > 0$ such that for every $n \geq n_0$,

$$\inf_{t\in[c_n,d_n]} \lambda_K(t)^2 \|\partial_t u(t)\|_{L^2}^2 \geq c_0. \tag{6.10}$$

If not, we can find a sequence $s_n \in [c_n, d_n]$ (after possibly passing to a subsequence) such that

$$\lim_{n\to\infty} \lambda_K(s_n)\|\partial_t u(s_n)\|_{L^2} = 0.$$

However, then Lemma 1.7 yields a sequence $r_n \to \infty$ such that, after passing to a further subsequence,

$$\lim_{n\to\infty} \delta_{r_n\lambda_K(s_n)}(u(s_n)) = 0.$$

Then Lemma 5.11 implies that

$$\lim_{n\to\infty} \mathbf{d}(s_n) = 0.$$

contradicting (6.7). As a consequence (6.10) holds. Therefore, using (6.10), (6.9), and (6.8) we have

$$\sum_{n\geq n_0} \int_{c_n}^{d_n} \|\partial_t u(t)\|_{L^2}^2 \,\mathrm{d}t \geq \frac{c_0}{4} \sum_{n\geq n_0} \int_{c_n}^{d_n} \lambda_K(c_n)^{-2} \,\mathrm{d}t \geq \frac{c_0}{4} \sum_{n\geq n_0} n^{-1} = \infty.$$

On the other hand, by (2.1) and the fact that the $[c_n, d_n]$ are disjoint, we have,

$$\sum_{n\geq n_0} \int_{c_n}^{d_n} \|\partial_t u(t)\|_{L^2}^2 \,\mathrm{d}t \leq \int_0^{T_*} \|\partial_t u(t)\|_{L^2}^2 \,\mathrm{d}t < \infty,$$

which is a contradiction. □

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